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Connections on locally trivial quantum principal fibre bundles

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Abstract

Budzyński and Kondracki [Rep. Math. Phys. 37 (1996) 365] have introduced a notion of locally trivial quantum principal fibre bundle making use of an algebraic notion of covering, which allows a reconstruction of the bundle from local pieces. Following this approach, we construct covariant differential algebras and connections on locally trivial quantum principal fibre bundles by gluing together such locally given geometric objects. We also consider covariant derivatives, connection forms, curvatures and curvature forms and explore the relations between these notions. As an example, a $U(1)$ quantum principal bundle over a glued quantum sphere as well as a connection in this bundle is constructed. The connection may be considered as a q-version of a Dirac monopole.

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1. Introduction

Since the appearance of quantum groups there has been a hope that it should be possible to use them instead of the classical symmetry groups of physical theories, in particular

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for quantum field theories. It was expected that the greater variety of group-like structures should lead, perhaps, to greater flexibility in the formulation of physical theories, thereby paving the way to a better understanding of fundamental problems of quantum theory and gravitation.

In (Lagrangian) quantum field theory, symmetry groups can be considered to appear in a very natural geometrical scheme. They are structure groups of principal fibre bundles. Moreover, on the classical level, all fields are geometrical objects living on the principal bundle or on associated fibre bundles. Thus, it is natural to ask for a generalization of the notion of principal bundle to a noncommutative situation. Thereby, in order to avoid unnecessary restrictions, one should replace not only the structure group by a quantum group, but also the base manifold (space–time) by a noncommutative space, which may even be necessary for physical reasons (see [9,10,13,15]).

In recent years, there have been several attempts to define such quantum principal bundles and the usual geometric objects that are needed to formulate gauge field theories on them, see [2,4,11,12,14,18,20,22,24]. Roughly following the same idea ("reversing the arrows"), the approaches differ in the details of the definitions. Closest to the classical idea that a locally trivial bundle should be imagined as being glued together from trivial pieces is the definition given in [4]. There, one starts with the notion of a covering of a quantum space. Being in the context of C^*-algebras, a covering is defined to be a (finite) family of closed ideals with zero intersection, which is easily seen to correspond to finite coverings by closed sets in the commutative case. C^*-algebras which have such a covering can be reconstructed from their "restriction" to the elements of the covering by a gluing procedure. Such a reconstruction is not always possible for general (not C^*-)algebras, as was noticed in [6]. The aim of [6] was to introduce differential calculi over algebras with covering. Leaving the C^*-category, one is confronted with the above difficulty, called "noncompleteness of a covering". Nevertheless, making use of "covering completions", if necessary, a general scheme for differential calculi on quantum spaces with covering was developed, and the example of the gluing of two quantum discs, being homeomorphic to the quantum sphere $S^2_{\mu c}$, $c > 0$, including the gluing of suitable differential calculi on the discs, was described in detail.

In [4], a locally trivial quantum principal fibre bundle having as base B such a quantum space with covering, and as fibre a compact quantum group H, is defined as a right H-comodule algebra with a covering adapted to the covering of the base. "Adapted" means that the ideals defining the covering appear as kernels of "locally trivializing" homomorphisms such that the intersections of these kernels with the embedded base are just the embeddings of the ideals defining the covering of B. Given such a locally trivial principal fibre bundle, one can define analogues of the classical transition functions which have the usual cocycle properties. Reversely, given such a cocycle one can reconstruct the bundle. The transition functions are algebra homomorphisms $H \to B_{ij}$, where B_{ij} is the algebra corresponding to the "overlap" of two elements of the covering of B. It turns out that they must have values in the centre of B_{ij}, which is related to the fact that principal bundles with structure group H are determined by bundles which have as structure group the classical subgroup of H, see [4].

The aim of the present paper is to introduce notions of differential geometry on locally trivial bundles in the sense of [4] in such a way that all objects can be glued together from local pieces.

Let us describe the contents of the paper. In Section 2, locally trivial principal bundles are defined slightly different from [4]. Not assuming C^*-algebras, we add to the definition of [4] the assumption that the "base" algebra is embedded as the algebra of right invariants into the "total space" algebra. This assumption has to be made in order to come back to the usual notion in the classical case, as is shown by an example. We prove a technical proposition about the restrictions of locally trivial principal bundles to overlaps of trivializations which in turn makes it possible to prove a reconstruction theorem for such bundles in terms of transition functions in the context of general algebras.

The aim of Section 3 is to introduce differential calculi on locally trivial quantum principal bundles. They are defined in such a way that they are uniquely determined by giving differential calculi on the "local pieces" of the base and a right-covariant differential calculus on the Hopf algebra (assuming that the calculi on the trivializations are graded tensor products). Uniqueness follows from the assumption that the local trivializing homomorphisms should be differentiable and that the kernels of their differential extensions should form a covering of the differential calculus on the total space, i.e. the differential calculus is "adapted" in the sense of [6]. This covering need not be complete. Thus, in order to have reconstructability, one has to use the covering completion, which in general is only a differential algebra.

Section 4 is the central part of the paper. Whereas in the classical situation there is a canonically given vertical part in the tangent space of a bundle, in the dual algebraic situation there is a canonically given horizontal subbimodule in the bimodule of forms of first degree on the bundle space. We start with the definition of left (right) covariant derivatives, which involves a Leibniz rule, a covariance condition, invariance of the submodule of horizontal forms, and a locality condition. Covariant derivatives can be characterized by families of linear maps $A_i : H \to \Gamma(B_i)$ satisfying $A_i(1) = 0$ and a compatibility condition being analogous to the classical relation between local connection forms. At this point a bigger differential algebra on the basis B appears, which is maximal among all the (LC) differential algebras being embeddable into the differential structure of the total space. Next we define left (right) connections as a choice of a projection of the left (right) $\mathcal{P}$-module of one-forms onto the submodule of horizontal forms being covariant under the right coaction and satisfying a locality condition. This is equivalent to the choice of a vertical complement to the submodule of horizontal forms. Left and right connections are equivalent. With this definition it is possible to reconstruct a connection from connections on the local pieces of the bundle. The corresponding linear maps $A_i : H \to \Gamma(B_i)$ satisfy the conditions for the A_i of covariant derivatives, and in addition $R \subset \ker A_i$ ($S^{-1}(R) \subset \ker A_i$), where R is the right ideal in H defining the right-covariant differential calculus there. Thus, connections are special cases of covariant derivatives. There is a corresponding notion of connection form as well as a corresponding notion of an exterior covariant derivative. The curvature can be defined as the square of the exterior covariant derivative, and is nicely related to a curvature form being defined by analogues of the structure equation. The local components of the curvature are related to the local connection forms in a nice way, and they are related among themselves by a homogeneous formula analogous to the classical one.

Finally, in Section 5, we give an example of a locally trivial principal bundle with a connection. The basis of the bundle, constructed in [6], is a *-algebra glued together from two copies of a quantum disc. The structure group is the classical group $U(1)$, and the bundle is defined by giving one transition function, which is sufficient because the covering of the basis has only two elements. Since all other coverings appearing in the example then have also two elements, there are no problems with noncomplete coverings. The differential calculus on the total space is determined by differential ideals in the universal differential calculi over the two copies of the quantum disc and the structure group. For the group, the ideal is chosen in a nonclassical way. Then, a connection is defined by giving explicitly two local connection forms. The curvature of this connection is nonzero.

In Appendix A, the relevant facts about coverings and gluings of algebras and differential algebras are collected, for the convenience of the reader. Details can be found in [6]. Moreover, we recall there some well-known facts about covariant differential calculi on quantum groups.

In the following, algebras are always assumed to be over $\mathbb{C}$, associative and unital. Ideals are assumed to be two-sided, up to some occasions, where their properties are explicitly specified. Tensor products of algebras are either equipped with the standard algebra structure (factorwise product) or, for differential algebras, with the $\mathbb{Z}_2$-graded product. If the words “homomorphism” and “isomorphism” are used for mappings of algebras, they mean homomorphisms and isomorphisms of algebras. The antipode of a Hopf algebra is always assumed to be invertible.

2. Locally trivial quantum principal fibre bundles

Following the ideas of [4], we introduce in this section the definition of a locally trivial quantum principal fibre bundle and prove propositions about the existence of trivial subbundles and about the reconstruction of the bundle. Essentially, this is contained in [4], up to some modifications: We do not assume C^*-algebras, and we add to the axioms the condition that the embedded base algebra coincides with the subalgebra of coinvariants. As structure group we take a general Hopf algebra.

In the sequel we use the results of [6], see also Appendix A. We recall here that, for an algebra B with a covering $(J_i)_{i\in I}$, there are canonical mappings $\pi_i : B \to B_i := B/J_i$, $\pi^i_j : B_i \to B_{ij} := B/(J_i + J_j)$, $\pi_{ij} : B \to B_{ij}$, etc.

Definition 1. A locally trivial quantum principal fibre bundle (QPFB) is a tupel

$$(\mathcal{P}, \Delta_{\mathcal{P}}, H, B, \iota, (\chi_i, J_i)_{i\in I}), \tag{1}$$

where B is an algebra, H is a Hopf algebra, $\mathcal{P}$ is a right H comodule algebra with coaction $\Delta_{\mathcal{P}}$, $(J_i)_{i\in I}$ is a complete covering of B, and χ_i and ι are homomorphisms with the following properties:

$$\begin{aligned}
&\chi_i : \mathcal{P} \to B_i \otimes H \ \text{ surjective}, && \iota : B \to \mathcal{P} \ \text{ injective},\\
&(id \otimes \Delta) \circ \chi_i = (\chi_i \otimes id) \circ \Delta_{\mathcal{P}}, && \chi_i \circ \iota(a) = \pi_i(a) \otimes 1, \quad a \in B,\\
&(\ker \chi_i)_{i\in I} \ \text{ complete covering of } \mathcal{P}, && \iota(B) = \{f \in \mathcal{P} | \Delta_{\mathcal{P}}(f) = f \otimes I\}.
\end{aligned}$$

Such a tupel we often denote simply by $\mathcal{P}$. Occasionally, $\mathcal{P}$, B and H are called total space, base space and structure group of the bundle.

The last assumption in Definition 1 does not appear in the definition of QPFB given in [4]. It is however used by other authors [1,11,20]. Already in the classical case this condition is needed to guarantee the transitive action of the structure group on the fibres, as shows the following example.

Example. Let M be a compact topological space covered by two closed subsets U_1 and U_2 being the closure of two open subsets covering M. Define $M_0 = U_1 \dot{\cup} U_2$ (disjoint union). M is obtained from M_0 identifying all corresponding points of U_1 and U_2. There is a natural projection $M_0 \to M$. Let us consider the algebras of continuous functions $C(M)$ and $C(M_0)$ over M and M_0, respectively. There exists an injective homomorphism $\kappa : C(M) \to C(M_0)$ being the pull back of the natural projection $M_0 \to M$. Suppose we have constructed a principal fibre bundle P over M_0 with structure group G, which is trivial on each of the disjoint components. Then we have an injective homomorphism $\iota_o : C(M_0) \to C(P)$ and two trivializations $\chi_{1,2} : C(P) \to C(U_{1,2}) \otimes C(G)$ with the properties assumed in Definition 1. The injective homomorphism $\iota : C(M) \to C(P)$, $\iota := \iota_0 \circ \kappa$, fulfills all the assumptions in Definition 1 up to the last one, and one obtains a fibration P over the base manifold M which is not a principal fibre bundle.

Proposition 1. *Let $\mathcal{P}_\mathrm{c}$ be the covering completion of $\mathcal{P}$ with respect to the complete covering* $(\ker \chi_i)_{i\in I}$. *Let $K : \mathcal{P} \to \mathcal{P}_\mathrm{c}$ be the corresponding isomorphism. The tupel*

$$(\mathcal{P}_\mathrm{c}, \Delta_{\mathcal{P}_\mathrm{c}}, H, B, \iota_\mathrm{c}, (\chi_{i_\mathrm{c}}, J_i)_{i\in I}),$$

where

$$\Delta_{\mathcal{P}_\mathrm{c}} = (K \otimes id) \circ \Delta_{\mathcal{P}} \circ K^{-1}, \qquad \chi_{i_\mathrm{c}} = \chi_i \circ K^{-1}, \qquad \iota_\mathrm{c} = K \circ \iota,$$

is a locally trivial QPFB.

The proof is obvious (transport of the structure using K).

Definition 2. A locally trivial QPFB $\mathcal{P}$ is called trivial if there exists an isomorphism $\chi : \mathcal{P} \to B \otimes H$ such that

$$\chi \circ \iota = id \otimes 1, \qquad (\chi \otimes id) \circ \Delta_{\mathcal{P}} = (id \otimes \Delta) \circ \chi.$$

Remark. A locally trivial QPFB with card $I = 1$, i.e. with trivial covering of B, is trivial. Triviality of the covering means that it consists of only one ideal $J = 0$. Moreover, there is only one trivializing epimorphism $\chi : \mathcal{P} \to B \otimes H$ which necessarily fulfills $\ker \chi = 0$.

There are several trivial QPFB related to a locally trivial QPFB. Define $\mathcal{P}_i := \mathcal{P}/\ker \chi_i$. Then $\tilde{\chi}_i : \mathcal{P}_i \to B_i \otimes H$ defined by

$$\tilde{\chi}_i(f + \ker \chi_i) := \chi_i(f) \tag{2}$$

is a well-defined isomorphism. $\iota_i : B_i \to \mathcal{P}_i$ defined by

$$\iota_i(b) := \tilde{\chi}_i^{-1}(b \otimes 1)$$

is injective and fulfills $\tilde{\chi}_i \circ \iota_i = id \otimes 1$. Moreover $\Delta_{\mathcal{P}_i} : \mathcal{P}_i \to \mathcal{P}_i \otimes H$ is well defined by

$$\Delta_{\mathcal{P}_i}(f + \ker \chi_i) := \Delta_{\mathcal{P}}(f) + \ker \chi_i \otimes H,$$

because from $(id \otimes \Delta) \circ \chi_i = (\chi_i \otimes id) \circ \Delta_{\mathcal{P}}$ follows $\Delta_{\mathcal{P}}(\ker \chi_i) \subset \ker \chi_i \otimes H$. Obviously, $\Delta_{\mathcal{P}_i}$ is a right coaction. Moreover, $(\tilde{\chi}_i \otimes id) \circ \Delta_{\mathcal{P}_i} = (id \otimes \Delta) \circ \tilde{\chi}_i$, and $\iota_i(B_i) = \{f \in \mathcal{P}_i | \Delta_{\mathcal{P}_i}(f) = f \otimes 1\}$. Thus $(\mathcal{P}_i, \Delta_{\mathcal{P}_i}, H, B_i, \iota_i, (\tilde{\chi}_i, 0))$ is a trivial QPFB.

Let $\mathcal{P}_{ij} := \mathcal{P}/(\ker \chi_i + \ker \chi_j)$. Then there is an isomorphism $\tilde{\chi}^i_{ij} : \mathcal{P}_{ij} \to (B_i \otimes H)/\chi_i(\ker \chi_j)$ given by

$$\tilde{\chi}^i_{ij}(f + \ker \chi_i + \ker \chi_j) := \chi_i(f) + \chi_i(\ker \chi_j). \tag{3}$$

It is natural to expect that P_{ij} should be a trivial bundle isomorphic to $B_{ij} \otimes H$. In fact, we will show that there is a natural isomorphism $(B_i \otimes H)/\chi_i(\ker \chi_j) \simeq B_{ij} \otimes H$, leading to trivialization maps $\chi^i_{ij} : \mathcal{P}_{ij} \to B_{ij} \otimes H$. Let us introduce the natural projections $\pi_{i_{\mathcal{P}}} : \mathcal{P} \to \mathcal{P}_i$, $\pi_{ij_{\mathcal{P}}} : \mathcal{P} \to \mathcal{P}_{ij}$ and $\pi^i_{j_{\mathcal{P}}} : \mathcal{P}_i \to \mathcal{P}_{ij}$. Obviously, $\tilde{\chi}_i \circ \pi_{i_{\mathcal{P}}} = \chi_i$, $\pi_{i_{\mathcal{P}}} = \tilde{\chi}_i^{-1} \circ \chi_i$ and $\pi_{ij_{\mathcal{P}}} = \pi^i_{j_{\mathcal{P}}} \circ \pi_{i_{\mathcal{P}}}$. We will need the following lemma, which generalizes an analogous lemma proved in [4] for the case of compact quantum groups.

Lemma 1. *Let B be an algebra and H be a Hopf algebra. Let $J \subset B \otimes H$ be an ideal with the property*

$$(id \otimes \Delta)J \subset J \otimes H.$$

Then there exists an ideal $I \subset B$ such that $J = I \otimes H$. This ideal is uniquely determined and equals $(id \otimes \varepsilon)(J)$.

Proof. It follows from surjectivity of $id \otimes \varepsilon$ that $I := (id \otimes \varepsilon)(J)$ is an ideal in B. We will show $J = I \otimes H$. First, we prove $J \subset I \otimes H$. Because of $(id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta) = id$ and $(id \otimes \Delta)J \subset J \otimes H$, we have $(id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta)J = J \subset I \otimes H$. $I \otimes H \subset J$ is a consequence of $I \otimes 1 \subset J$, which is proved as follows: A general element of I has the form $\sum_k a_k \varepsilon(h_k)$, where $\sum_k a_k \otimes h_k \in J$. Because of

$$\sum_k a_k \varepsilon(h_k) \otimes 1 = \sum_k \sum (a_k \otimes h_{k_{(1)}})(1 \otimes S(h_{k_{(2)}}))$$

and

$$(id \otimes \Delta)\left(\sum_k a_k \otimes h_k\right) = \sum_k a_k \otimes h_{k_{(1)}} \otimes h_{k_{(2)}} \in J \otimes H,$$

$\sum_k a_k \varepsilon(h_k) \otimes 1$ is an element of J. □

Proposition 2. *$\mathcal{P}_{ij}$ is a trivial QPFB, i.e. there exist*

$$\chi^i_{ij} : \mathcal{P}_{ij} \to B_{ij} \otimes H, \qquad \Delta_{\mathcal{P}_{ij}} : \mathcal{P}_{ij} \to \mathcal{P}_{ij} \otimes H, \qquad \iota_{ij} : B_{ij} \to \mathcal{P}_{ij},$$

such that the conditions of Definitions 1 and 2 *are satisfied.*

Remark. $\mathcal{P}_{ij}$ is a trivial QPFB in two ways by choosing χ^i_{ij} or χ^j_{ij}. The composition of these maps just gives the transition functions.

Proof. Applying $\chi_i \otimes id$ to $\Delta_{\mathcal{P}}(\ker \chi_j) \subset \ker \chi_j \otimes H$ and using $(id \otimes \Delta) \circ \chi_i = (\chi_i \otimes id) \circ \Delta_{\mathcal{P}}$ it follows that $(id \otimes \Delta) \circ \chi_i(\ker \chi_j) \subset \chi_i(\ker \chi_j) \otimes H$. By Lemma 1, there exist ideals $\tilde{K}^i_j \subset B_i$ such that $\chi_i(\ker \chi_j) = \tilde{K}^i_j \otimes H$. $K^i_j := \pi^i_j(\tilde{K}^i_j)$ is an ideal in B_{ij}. Our aim is to show $\tilde{K}^i_j = \pi_i(J_j)$, because then we have a natural isomorphism $B_i \otimes H/\chi_i(\ker \chi_j) \simeq B_{ij} \otimes H$ whose composition with $\tilde{\chi}^i_{ij}$ gives the desired χ^i_{ij}.

First we show $\pi_i(J_j) \subset \tilde{K}^i_j$. According to Lemma 1, we have $\tilde{K}^i_j = (id \otimes \varepsilon)(\chi_i(\ker \chi_j))$. We need to show that for $b \in J_j$ there exists $\tilde{b} \in \ker \chi_j$ with $(id \otimes \varepsilon) \circ \chi_i(\tilde{b}) = \pi_i(b)$. This is obviously achieved by taking $\tilde{b} = \iota(b)$.

Using this inclusion, one finds that there is a canonical isomorphism $(B_i \otimes H)/\chi_i(\ker \chi_j) \simeq (B_{ij}/K^i_j) \otimes H$ given by $b \otimes h + \chi_i(\ker \chi_j) \to (\pi_i(b) + K^i_j) \otimes h$. Composing with $\tilde{\chi}^i_{ij}$ (see (3)), there results an isomorphism $\chi^i_{ij} : \mathcal{P}_{ij} \to B_{ij}/K^i_j \otimes H$ given by

$$\chi^i_{ij}(f + \ker \chi_i + \ker \chi_j) := (\pi^i_j \otimes id) \circ \chi_i(f) + K^i_j \otimes H.$$

Our goal is now to show $K^i_j = \pi^i_j(\tilde{K}^i_j) = 0$.

As a first step we will prove $K^i_j = K^j_i$. To this end, we note that

$$\tilde{\phi}_{ji} := \chi^j_{ij} \circ \pi^i_{j\mathcal{P}} \circ \tilde{\chi}_i^{-1}$$

is a homomorphism $\tilde{\phi}_{ji} : B_i \otimes H \to B_{ij}/K^j_i \otimes H$ with $\ker \tilde{\phi}_{ji} = \tilde{K}^i_j \otimes H$. In terms of this homomorphism, we define a homomorphism $\psi_{ji} : B_{ij} \to B_{ij}/K^j_i$ by

$$\psi_{ji}(a + J_i + J_j) := (id \otimes \varepsilon) \circ \tilde{\phi}_{ji}((a + J_i) \otimes 1).$$

ψ_{ji} is well defined due to the inclusion $\pi_i(J_j) \subset \tilde{K}^i_j$ already proved above. From $\ker \tilde{\phi}_{ji} = \tilde{K}^i_j \otimes H$ easily follows that $\ker \psi_{ji} \supset K^i_j$. On the other hand, the following calculation shows that $\psi_{ji} : B_{ij} \to B_{ij}/K^j_i$ is the natural projection, and therefore $K^i_j = K^j_i$

$$\begin{aligned}
&\psi_{ji}(a + J_i + J_j)\\
&\quad = (id \otimes \varepsilon) \circ \tilde{\phi}_{ji}((a + J_i) \otimes 1) = (id \otimes \varepsilon) \circ \chi^j_{ij} \circ \pi^i_{j\mathcal{P}} \circ \tilde{\chi}_i^{-1}((a + J_i) \otimes 1)\\
&\quad = (id \otimes \varepsilon) \circ \chi^j_{ij} \circ \pi_{ij\mathcal{P}}(\iota(a)) = (id \otimes \varepsilon) \circ \chi^j_{ij}(\iota(a) + \ker \chi_i + \ker \chi_j)\\
&\quad = (id \otimes \varepsilon)((\pi^j_i \otimes id) \circ \chi_j(\iota(a)) + K^j_i \otimes H)\\
&\quad = (id \otimes \varepsilon)((\pi^j_i \otimes id)(\pi_j(a) \otimes 1) + K^j_i \otimes H)\\
&\quad = \pi_{ij}(a) + K^j_i.
\end{aligned}$$

Note that $K^i_j = K^j_i$ also means $\psi_{ij} = \psi_{ji}$.

For showing $K^i_j = 0$, we use the completeness of the covering $(\ker \chi_i)_{i\in I}$. The covering completion of $\mathcal{P}$ is by definition

$$\mathcal{P}_\mathrm{c} = \left\{ (f_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{P}/\ker \chi_i | \pi^i_{j_\mathcal{P}}(f_i) = \pi^j_{i_\mathcal{P}}(f_j) \right\}.$$

We introduce a locally trivial QPFB $\breve{\mathcal{P}} \simeq \mathcal{P}_\mathrm{c}$ such that a comparison of $\breve{\mathcal{P}}^{coH} = \{f \in \breve{\mathcal{P}} | \Delta_{\breve{\mathcal{P}}}(f) = f \otimes 1\}$ with $B \simeq B_\mathrm{c}$ allows to read off $\ker \psi_{ij} = K^i_j = 0$. Let $\phi_{ij} : B_{ij}/K^i_j \otimes H \to B_{ij}/K^i_j \otimes H$ be the isomorphisms defined by

$$\phi_{ij} := \chi^i_{ij} \circ \chi^{j\,-1}_{ij}.$$

Using the identities

$$\chi^i_{ij} \circ \pi^i_{j_\mathcal{P}} \circ \tilde{\chi}_i^{-1} = (\psi_{ij} \otimes id) \circ (\pi^i_j \otimes id)$$

it is easy to verify that the algebra $\mathcal{P}_\mathrm{c}$ is isomorphic to the algebra

$$\breve{\mathcal{P}} = \left\{ (g_i)_{i\in I} \in \bigoplus_{i\in I} (B_i \otimes H) \middle| (\psi_{ji} \otimes id) \circ (\pi^i_j \otimes id)(g_i) \right. \\ \left. = \phi_{ij} \circ (\psi_{ji} \otimes id) \circ (\pi^j_i \otimes id)(g_j) \right\} \tag{4}$$

(cf. Lemma 1 in [6]), and the corresponding isomorphism $\chi : \mathcal{P}_\mathrm{c} \to \breve{\mathcal{P}}$ is defined by $\chi((f_i)_{i\in I}) := (\tilde{\chi}_i(f_i))_{i\in I}$. Transporting the homomorphisms $\Delta_{\mathcal{P}_\mathrm{c}}$, χ_{i_c} and ι_c to $\Delta_{\breve{\mathcal{P}}} := (\chi \otimes id) \circ \Delta_{\mathcal{P}_\mathrm{c}} \circ \chi^{-1}$, $\breve{\chi}_i := \chi_{i_\mathrm{c}} \circ \chi^{-1}$ and $\breve{\iota} := \chi \circ \iota_\mathrm{c}$ respectively, one obtains a locally trivial QPFB again. Explicitly,

$$\Delta_{\breve{\mathcal{P}}}((g_i)_{i\in I}) = ((id \otimes \Delta)(g_i))_{i\in I}, \qquad \breve{\chi}_i((g_k)_{k\in I}) = g_i, \qquad \breve{\iota}(a) = (\pi_i(a) \otimes 1)_{i\in I}.$$

Using the existence of $\Delta_{\breve{P}}$ and $\breve{\iota}$, and surjectivity of ψ_{ji} and π^i_j, one easily shows that the isomorphisms ϕ_{ij} fulfill

$$(id \otimes \Delta) \circ \phi_{ij} = (\phi_{ij} \otimes id) \circ (id \otimes \Delta), \tag{5}$$

$$\phi_{ij}(a \otimes 1) = a \otimes 1, \quad a \in B_{ij}/K^i_j. \tag{6}$$

Using (5) and (6) it follows that the subalgebra $\breve{\mathcal{P}}^{coH} = \{f \in \breve{\mathcal{P}} | \Delta_{\breve{\mathcal{P}}}(f) = f \otimes 1\}$ is isomorphic to

$$\breve{\mathcal{P}}^{coH} = \left\{ (a_i \otimes 1)_{i\in I} \in \bigoplus_{i\in I} B_i \otimes 1 \middle| \psi_{ji} \circ \pi^i_j(a_i) \otimes 1 = \psi_{ji} \circ \pi^j_i(a_j) \otimes 1 \right\}.$$

This algebra is by Definition 1 isomorphic to

$$B \simeq B_\mathrm{c} = \left\{ (a_i)_{i\in I} \in \bigoplus_{i\in I} B_i \middle| \pi^i_j(a_i) = \pi^j_i(a_j) \right\}$$

(see [6] and Appendix A). It follows that the ψ_{ij} have to be isomorphisms, i.e. $\ker \psi_{ij} = K^i_j = 0$, which means in fact $\psi_{ij} = id$. Thus, $\chi^i_{ij} : \mathcal{P}_{ij} \to B_{ij} \otimes H$ are isomorphisms.

Further define $\Delta_{\mathcal{P}_{ij}} : \mathcal{P}_{ij} \to \mathcal{P}_{ij} \otimes H$ by

$$\Delta_{\mathcal{P}_{ij}}(f + \ker \chi_i + \ker \chi_j) := \Delta_{\mathcal{P}}(f) + (\ker \chi_i + \ker \chi_j) \otimes H$$

and $\iota_{ij} : B_{ij} \to \mathcal{P}_{ij}$ by

$$\iota_{ij}(\pi_{ij}(a)) := \iota(a) + \ker \chi_i + \ker \chi_j.$$

It is easy to verify that all the conditions of Definition 2 are satisfied. □

Notice that, due to $K^i_j = 0$, we have $\tilde{K}^i_j = \pi_i(J_j)$. This means

$$\chi_i(\ker \chi_j) = \pi_i(J_j) \otimes H. \tag{7}$$

The isomorphisms χ^i_{ij} satisfy

$$\chi^i_{ij} \circ \pi_{ij\mathcal{P}} = (\pi^i_j \otimes id) \circ \chi_i, \tag{8}$$

and the ϕ_{ij} defined above are isomorphisms $B_{ij} \otimes H \to B_{ij} \otimes H$ fulfilling (5) and (6) $\phi_{ij} \circ \phi_{ji} = id$.

Notice that the isomorphism K appearing in Proposition 1 is a bundle isomorphism in the following sense:

Definition 3. Two locally trivial QPFBs $(\mathcal{P}, \Delta_{\mathcal{P}}, H, B, \iota, (\chi_i, J_i)_{i\in I})$ and $(\mathcal{P}', \Delta_{\mathcal{P}'}, H, B, \iota', (\chi'_i, J_i)_{i\in I})$ with the same structure group H, the same base B and the same covering $(J_i)_{i\in I}$ of B are said to be isomorphic, if there exists an isomorphism $\psi : \mathcal{P} \to \mathcal{P}'$ such that

$$\psi \circ \iota = \iota', \tag{9}$$

$$(\psi \otimes id) \circ \Delta_{\mathcal{P}} = \Delta_{\mathcal{P}'} \circ \psi, \tag{10}$$

$$\psi(\ker \chi_i) = \ker \chi'_i. \tag{11}$$

For classical locally trivial principal fibre bundles the condition (11) follows from (9) (a bundle isomorphism preserves the fibres). $\ker \chi_i$ is in this case the set of functions vanishing on some trivialized piece of the bundle space. We were not able to derive (11) from the other assumptions in general. However, condition (11) is fulfilled automatically in the case $I = \{1, 2\}$. This follows from the following proposition.

Proposition 3. *For a locally trivial QPFB* $(\mathcal{P}, \Delta_{\mathcal{P}}, H, B, \iota, (\chi_i, J_i)_{i\in\{1,2\}})$ *the equalities*

$$\ker \chi_i = \mathcal{P}\iota(J_i)\mathcal{P}, \quad i = 1, 2, \tag{12}$$

are valid.

Proof. The inclusion $\ker \chi_i \supset \mathcal{P}\iota(J_i)\mathcal{P}$ is obvious from $\chi \circ \iota = \pi_i \otimes 1_H$. To prove the other inclusion, assume $\chi_1(p) = 0$. It follows from (8) that $(\pi^1_2 \otimes id) \circ \chi_1(p) = \phi_{12} \circ (\pi^2_1 \otimes id) \circ \chi_2(p) = 0$, therefore $\chi_2(p) \in \ker(\pi^2_1 \otimes id) = \pi_2(J_1) \otimes H$. Thus, one finds $b_k \in J_1$ and $h_k \in H$ such that $\chi_2(p) = \sum_k \pi_2(b_k) \otimes h_k$. Due to the surjectivity of

χ_2 there exists $\tilde{h}_k \in \mathcal{P}$ such that $\chi_2(\tilde{h}_k) = 1 \otimes h_k$. The element $p_2 := \sum_k \iota(b_k)\tilde{h}_k$ is in $\mathcal{P}\iota(J_1)\mathcal{P} \subset \ker \chi_1$ and therefore $p - p_2 \in \ker \chi_1$. But $p - p_2 \in \ker \chi_2$ by construction, and thus $p - p_2 \in \ker \chi_1 \cap \ker \chi_2 = \{0\}$. □

Eq. (11) is in the case $I = \{1, 2\}$ an immediate consequence of this proposition

$$\psi(\ker \chi_i) = \mathcal{P}'\psi(\iota(J_i))\mathcal{P}' = \mathcal{P}'\iota'(J_i)\mathcal{P}' = \ker \chi_i'. \tag{13}$$

Proposition 4 (cf. Budzyński and W. Kondracki [4]). *A locally trivial QPFB over a basis B with complete covering $(J_i)_{i\in I}$ and with structure group H defines a family of homomorphisms*

$$\tau_{ij} : H \to B_{ij}$$

called transition functions, satisfying the conditions

$$\tau_{ii}(h) = 1\varepsilon(h) \quad \forall h \in H, \qquad \tau_{ji}(S(h)) = \tau_{ij}(h) \quad \forall h \in H,$$
$$\tau_{ij}(h)a = a\tau_{ij}(h) \quad \forall a \in B_{ij}, \quad h \in H,$$
$$\pi_k^{ij} \circ \tau_{ij}(h) = m_{B_{ijk}} \circ ((\pi_j^{ik} \circ \tau_{ik}) \otimes (\pi_i^{jk} \circ \tau_{kj})) \circ \Delta(h) \quad \forall h \in H.$$

Here, $\pi_j^{ik} : B_{ik} \to B_{ijk}$ are the canonical homomorphisms, $m_{B_{ijk}}$ is the multiplication in B_{ijk}. On the other hand, every family of transition functions (i.e. of homomorphisms τ_{ij} with the above properties) related to an algebra B with complete covering $(J_i)_{i\in I}$ and a Hopf algebra H determines a locally trivial QPFB $(\mathcal{P}_\tau, \Delta_{\mathcal{P}_\tau}, H, B, \iota_\tau, (\chi_{i_\tau}, J_i)_{i\in I})$. If the transition functions stem from a given locally trivial QPFB $(\mathcal{P}, \Delta_{\mathcal{P}}, H, B, \iota, (\chi_i, J_i)_{i\in I})$, the bundle $\mathcal{P}_\tau$ is isomorphic to $\mathcal{P}$.

Proof. Let a bundle $\mathcal{P}$ be given and let the $\phi_{ij} : B_{ij} \otimes H \to B_{ij} \otimes H$ be defined as above. Define homomorphisms $\tau_{ij} : H \to B_{ij}$ by

$$\tau_{ji}(h) := (id \otimes \varepsilon) \circ \phi_{ij}(1 \otimes h). \tag{14}$$

(There is another possible choice, $\tau_{ij}(h) := (id\otimes\varepsilon)\phi_{ij}(1\otimes h)$, which corresponds to another form of the cocycle condition.) One shows that (14) is equivalent to

$$\phi_{ij}(a \otimes h) = \sum a\tau_{ji}(h_{(1)}) \otimes h_{(2)}. \tag{15}$$

Using (5), (6) and $(\varepsilon \otimes id) \circ \Delta = id$ it follows from (14) that

$$\begin{aligned}\sum a\tau_{ji}(h_{(1)}) \otimes h_{(2)} &= \sum (a \otimes 1)((id \otimes \varepsilon) \circ \phi_{ij}(1 \otimes h_{(1)}) \otimes h_{(2)})\\ &= (a \otimes 1)(id \otimes \varepsilon \otimes id) \circ (\phi_{ij} \otimes id) \circ (id \otimes \Delta)(1 \otimes h)\\ &= (a \otimes 1)(id \otimes \varepsilon \otimes id) \circ (id \otimes \Delta) \circ \phi_{ij}(1 \otimes h)\\ &= (a \otimes 1)\phi_{ij}(1 \otimes h) = \phi_{ij}(a \otimes h).\end{aligned}$$

Conversely, if (15) is satisfied, the choice $a = 1$, together with $\varepsilon(h_{(2)})\otimes h_{(1)} = h$ gives (14). $\tau_{ii}(h) = \varepsilon(h)1$ follows from $\phi_{ii} = id$. Every homomorphism $\tau_{ij} : H \to B$ is convolution invertible with convolution inverse $\tau_{ij}^{-1} = \tau_{ij} \circ S$. On the other hand from $\phi_{ij} \circ \phi_{ji} = id$ easily follows $\tau_{ij}^{-1} = \tau_{ji}$:

$$\phi_{ij} \circ \phi_{ji}(1 \otimes h) = \phi_{ij}\left(\sum \tau_{ij}(h_{(1)}) \otimes h_{(2)}\right) = \sum \tau_{ij}(h_{(1)})\tau_{ji}(h_{(2)}) \otimes h_{(3)} = 1 \otimes h.$$

Applying $id \otimes \varepsilon$, we obtain $\sum \tau_{ij}(h_{(1)})\tau_{ji}(h_{(2)}) = \varepsilon(h)1$, i.e. $\tau_{ji} = \tau_{ij} \circ S$. τ_{ij} has values in the centre of B_{ij}

$$\begin{aligned} a\tau_{ij}(h) - \tau_{ij}(h)a &= a(id \otimes \varepsilon)\phi_{ji}(1 \otimes h) - (id \otimes \varepsilon)\phi_{ji}(1 \otimes h)a \\ &= (id \otimes \varepsilon)((a \otimes 1)\phi_{ji}(1 \otimes h) - \phi_{ji}(1 \otimes h)(a \otimes 1)) \\ &= (id \otimes \varepsilon)(\phi_{ji}((a \otimes h) - (a \otimes h))) = 0. \end{aligned}$$

To prove the last relation of the proposition, define isomorphisms $\phi_{ij}^k : B_{ijk} \otimes H \to B_{ijk} \otimes H$ by

$$\phi_{ij}^k((a \otimes h) + \pi_{ij}(J_k) \otimes H) := \phi_{ij}(a \otimes h) + \pi_{ij}(J_k) \otimes H$$

(using $B_{ijk} \simeq B_{ij}/\pi_{ij}(J_k)$). ϕ_{ij}^k are well defined because of $\phi_{ij}(a \otimes 1) = a \otimes 1$. Now, a lengthy but simple computation leads to

$$\phi_{ij}^k = \phi_{ik}^j \circ \phi_{kj}^i.$$

The idea of this computation is to consider the isomorphism $\chi_i^{ijk} : \mathcal{P}/(\ker \chi_i + \ker \chi_j + \ker \chi_k) \to B_{ijk} \otimes H$ induced by χ_i and to prove $\phi_{ij}^k = \chi_i^{ijk} \circ \chi_j^{ijk-1}$.

Combining the definition of ϕ_{ij}^k with (15), one obtains

$$\phi_{ij}^k(a \otimes h) = \sum a\pi_k^{ij} \circ \tau_{ji}(h_{(1)}) \otimes h_{(2)}.$$

Therefore, taking $a = 1$ and applying $id \otimes \varepsilon$,

$$\pi_k^{ij} \circ \tau_{ji}(h) = (id \otimes \varepsilon) \circ \phi_{ij}^k(1 \otimes h).$$

Inserting here $\phi_{ij}^k(1 \otimes h) = \phi_{ik}^j \circ \phi_{kj}^i(1 \otimes h)$ yields

$$\pi_k^{ij} \circ \tau_{ji}(h) = \sum (\pi_i^{kj} \circ \tau_{jk}(h_{(1)}))(\pi_j^{ik} \circ \tau_{ki}(h_{(2)})).$$

This ends the proof of one direction of the proposition.

We will not give the details of reconstruction of the bundle from the transition functions. We only remark that, for a given family of transition functions τ_{ij}, we define the isomorphisms ϕ_{ij} by formula (15), which gives rise to the gluing

$$\mathcal{P}_\tau = \left\{ (f_i)_{i\in I} \in \bigoplus_{i\in I}(B_i \otimes H) \,\middle|\, (\pi_j^i \otimes id)(f_i) = \phi_{ij} \circ (\pi_i^j \otimes id)(f_j) \right\}. \tag{16}$$

One verifies that the formulas

$$\Delta_{\mathcal{P}_\tau}((f_i)_{i\in I}) = (id \otimes \Delta(f_i))_{i\in I} \quad \forall (f_i)_{i\in I} \in \mathcal{P}_\tau, \tag{17}$$

$$\chi_{\tau k}((f_i)_{i\in I}) = f_k \quad \forall (f_i)_{i\in I} \in \mathcal{P}_\tau, \tag{18}$$

$$\iota_\tau(a) = (\pi_i(a)\otimes 1)_{i\in I} \quad \forall a \in B \tag{19}$$

define a locally trivial QPFB $(\mathcal{P}_\tau, \Delta_{\mathcal{P}_\tau}, H, B, \iota_\tau, (\chi_{\tau i}, J_i)_{i\in I})$. If the τ_{ij} stem from a given locally trivial QPFB $\mathcal{P}$, applying the isomorphism χ^{-1} defined as above (proof of Proposition 2) leads to $\mathcal{P}_c \simeq \mathcal{P}_\tau$. □

Note that the QPFBs $\mathcal{P}, \mathcal{P}_c, \mathcal{P}_\tau$ have identical transition functions.

The following proposition summarizes Definition 9, Proposition 4 and Theorem 3 of [4].

Proposition 5. *Families (τ_{ij}) and $({\tau'}_{ij})$ of transition functions related to the same covering of the base B and the same Hopf algebra H define isomorphic locally trivial QPFBs if and only if there exists a family $(\sigma_i)_{i\in I}$ of homomorphisms $\sigma_i : H \to B_i$ with values in the centre of B_i such that*

$$\tau'_{ij} = m_{B_{ij}} \circ (m_{B_{ij}} \otimes id) \circ (\pi^i_j \otimes id \otimes pi^j_i) \circ ((\sigma_i \circ S) \otimes \tau_{ij} \otimes \sigma_j) \circ (id \otimes \Delta) \circ \Delta, \tag{20}$$

or, in Sweedler notation,

$$\tau'_{ij}(h) = \sum (\pi^i_j \circ \sigma_i \circ S(h_{(1)}))(\tau_{ij}(h_{(2)}))(\pi^j_i \circ \sigma_j(h_{(3)})). \tag{21}$$

Proof. Using the isomorphy to gluings of the type (16), the proof relies on the following. First, a bundle isomorphism ψ in the sense of Definition 3 gives rise to isomorphisms $\psi_i : B_i \otimes H \to B_i \otimes H$ of trivial bundles by

$$\psi_i \circ \chi_i = \chi'_i \circ \psi. \tag{22}$$

The ψ_i are bijective due to (11). Corresponding homomorphisms $\sigma_i : H \to B_i$ are defined by

$$\sigma_i(h) = (id \otimes \varepsilon) \circ \psi_i(1 \otimes h). \tag{23}$$

On the other hand, if a family of centre-valued homomorphisms $\sigma_i : H \to B_i$ is given, corresponding isomorphisms $\psi_i : B_i \otimes H \to B_i \otimes H$ are defined by

$$\psi_i(b \otimes h) = \sum b\sigma_i(h_{(1)}) \otimes h_{(2)}. \tag{24}$$

We leave the details of the argument to the reader. □

3. Adapted covariant differential structures on locally trivial QPFB

In the sequel, we will use the skew tensor product of differential calculi. Let $\Gamma(A)$ and $\Gamma(B)$ be two differential calculi. We define the differential calculus $\Gamma(A)\hat{\otimes}\Gamma(B)$ as the vector space $\Gamma(A) \otimes \Gamma(B)$ equipped with the product

$$(\gamma\hat{\otimes}\rho)(\omega\hat{\otimes}\tau) = (-1)^{mn}(\gamma\omega\hat{\otimes}\rho\tau),$$
$$\omega \in \Gamma^n(A),\quad \rho \in \Gamma^m(B),\quad \gamma \in \Gamma(A),\quad \tau \in \Gamma(B) \tag{25}$$

and the differential

$$d(\gamma\hat{\otimes}\rho) = (d\gamma\hat{\otimes}\rho) + (-1)^n(\gamma\hat{\otimes}d\rho), \quad \gamma \in \Gamma^n(A), \ \ \rho \in \Gamma(B). \tag{26}$$

We remind the reader that for every differential calculus $\Gamma(A)$ there exists a differential ideal $J(A)$ in the universal differential calculus $\Omega(A)$ such that $\Gamma(A) \simeq \Omega(A)/J(A)$. There may be different $J(A)$ leading to isomorphic $\Omega(A)/J(A)$. We always choose $J(A) = \ker(id_{A_{\Omega\to\Gamma}})$ (see Appendix A for this notation) and call this ideal the differential ideal corresponding to $\Gamma(A)$.

Proposition 6. *Let $\Gamma(A)$ and $\Gamma(B)$ be two differential calculi and let $J(A) \subset \Omega(A)$ and $J(B) \subset \Omega(B)$ be the corresponding differential ideals, respectively. Let $id\otimes 1 : A \to A\otimes B$ and $1 \otimes id : B \to A \otimes B$ be the embedding homomorphisms. Then, the differential ideal $J(A \otimes B) \subset \Omega(A \otimes B)$ corresponding to $\Gamma(A)\hat{\otimes}\Gamma(B)$ is generated by the sets*

$$(id \otimes 1)_\Omega(J(A)); \ (1 \otimes id)_\Omega(J(B))$$
$$\{(a \otimes 1)d(1 \otimes b) - (d(1 \otimes b))(a \otimes 1) | a \in A, \ b \in B\}. \tag{27}$$

Proof. $J(A \otimes B)$ is defined as the kernel of $\tilde{\psi} := (id_{A\otimes B})_{\Omega\to\Gamma} : \Omega(A \otimes B) \to \Gamma(A)\hat{\otimes}\Gamma(B)$. $\tilde{\psi}$ can be written explicitly as follows:

$$\tilde{\psi}\left(\sum_k (a_k^0 \otimes 1)d(a_k^1 \otimes 1)\right) = \sum_k a_k^0 da_k^1 \hat{\otimes} 1,$$

$$\tilde{\psi}\left(\sum_k (1 \otimes b_k^0)d(1 \otimes b_k^1)\right) = \sum_k 1\hat{\otimes} b_k^0 db_k^1.$$

Using the rules (25) and (26) it is easy to verify that the differential ideal $\tilde{J}(A\otimes B)$ generated by the sets (27) satisfies $\tilde{J}(A \otimes B) \subset \ker\tilde{\psi}$. Let $\tilde{\Gamma}(A\otimes B) = \Omega(A\otimes B)/\tilde{J}(A\otimes B)$. Using the Leibniz rule and $d^2 = 0$, one concludes easily from (27) that there are the following relations in $\tilde{\Gamma}(A \otimes B)$:

$$(d(a \otimes 1))(1 \otimes b) = (1 \otimes b)d(a \otimes 1), \tag{28}$$

$$d(a \otimes 1)d(1 \otimes b) = -d(1 \otimes b)d(a \otimes 1). \tag{29}$$

We define now a homomorphism $\psi : \tilde{\Gamma}(A \otimes B) \to \Gamma(A)\hat{\otimes}\Gamma(B)$ by $\psi \circ \pi = \tilde{\psi}$, where $\pi : \Omega(A \otimes B) \to \tilde{\Gamma}(A \otimes B)$ is the quotient map with respect to $\tilde{J}(A \otimes B)$. ψ is well defined because of $\tilde{J}(A \otimes B) \subset \ker\tilde{\psi}$. On the other hand, we define a linear map $\phi : \Gamma(A)\hat{\otimes}\Gamma(B) \to \tilde{\Gamma}(A \otimes B)$. Due to $(id \otimes 1_B)_\Omega(J(A)) \subset \tilde{J}(A \otimes B)$ and $(1_A \otimes id)_\Omega(J(B)) \subset \tilde{J}(A\otimes B)$ there exist homomorphisms $\Upsilon_A := (id\otimes 1_B)_\Gamma : \Gamma(A) \to \tilde{\Gamma}(A \otimes B)$ and $\Upsilon_B := (1_A \otimes id)_\Gamma : \Gamma(B) \to \tilde{\Gamma}(A \otimes B)$, and ϕ is defined by

$$\phi(\alpha\hat{\otimes}\beta) := \Upsilon_A(\alpha)\Upsilon_B(\beta).$$

ϕ is well defined due to the universality of the tensor product $\Gamma(A) \otimes \Gamma(B)$. Now, a direct computation, making use of (28) and (29), shows that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. Thus, ψ is an isomorphism, and it follows that $J(A \otimes B) = \ker\tilde{\psi} = \tilde{J}(A \otimes B)$. □

Remark. If we are in a converse situation, i.e. if a differential calculus $\Gamma(A \otimes B)$ with corresponding differential ideal $J(A \otimes B)$ is given, there exist differential ideals $J(A) := J(A\otimes B)\cap\Omega(A\otimes 1)$ and $J(B) := J(A\otimes B)\cap\Omega(1\otimes B)$. By Proposition 6, the differential calculus is isomorphic to an algebra of the form $\Gamma(A)\hat{\otimes}\Gamma(B)$ if and only if $J(A \otimes B)$ is generated by the sets (27).

In the sequel, we always identify $\mathcal{P}/\ker\chi_i$ with $B_i \otimes H$, by means of the isomorphisms $\tilde{\chi}_i$ (see (2)).

Our goal is now to define differential structures on $\mathcal{P}$. By Proposition 22, a family of differential calculi $\Gamma(B_i)$ and a right-covariant differential calculus $\Gamma(H)$ determine unique differential calculi $\Gamma(B)$ and $\Gamma(\mathcal{P})$ such that $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ and $(\Gamma(\mathcal{P}), (\Gamma(B_i)\hat{\otimes}\Gamma(H))_{i\in I}$ are adapted to $(B, (J_i)_{i\in I})$ and $(\mathcal{P}, (\ker\chi_i)_{i\in I})$, respectively. $\Gamma(\mathcal{P})$ and $\Gamma(B)$ are given in the following way: one has the extensions $\chi_{i_{\Omega\to\Gamma}} : \Omega(\mathcal{P}) \to \Gamma(B_i)\hat{\otimes}\Gamma(H)$ and $\pi_{i_{\Omega\to\Gamma}} : \Omega(B) \to \Gamma(B_i)$ of the χ_i and π_i, respectively. These extensions give rise to differential ideals $\ker\chi_{i_{\Omega\to\Gamma}} \subset \Omega(\mathcal{P})$ and $\ker\pi_{i_{\Omega\to\Gamma}} \subset \Omega(B)$, thus $J(\mathcal{P}) := \cap_{i\in I}\ker\chi_{i_{\Omega\to\Gamma}}$ and $J(B) := \cap_{i\in I}\ker\pi_{i_{\Omega\to\Gamma}}$ are differential ideals. By construction, $\Gamma(\mathcal{P}) := \Omega(\mathcal{P})/J(\mathcal{P})$ and $\Gamma(B) := \Omega(B)/J(B)$ are adapted, i.e. the extensions $\chi_{i_\Gamma} : \Gamma(\mathcal{P}) \to \Gamma(B_i)\hat{\otimes}\Gamma(H)$ and $\pi_{i_\Gamma} : \Gamma(B) \to \Gamma(B_i)$ of the χ_i and π_i exist and fulfill $\cap_{i\in I}\ker\chi_{i_\Gamma} = 0$ and $\cap_{i\in I}\ker\pi_{i_\Gamma} = 0$, respectively.

Definition 4. A differential structure on a locally trivial QPFB is a differential calculus $\Gamma(\mathcal{P})$ defined by a family of differential calculi $\Gamma(B_i)$ and a right-covariant differential calculus $\Gamma(H)$, as described above.

Proposition 7. *Let $\Gamma(\mathcal{P})$ be a differential structure on $\mathcal{P}$, and let $\Gamma(B)$ be determined by the corresponding $\Gamma(B_i)$ as above. Then $\Gamma(\mathcal{P})$ is covariant, i.e. there exists a right coaction $\Delta^\Gamma_{\mathcal{P}} : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P})\otimes H$ extending $\Delta_{\mathcal{P}}$ and being compatibel with d, $\Delta^\Gamma_{\mathcal{P}}(df) = (d \otimes id) \circ \Delta_{\mathcal{P}}(f)$ (cf. Definition* 13). *The χ_{i_Γ} satisfy*

$$\Delta^\Gamma_{\mathcal{P}}(\ker\chi_{i_\Gamma}) \subset \ker\chi_{i_\Gamma} \otimes H \quad \forall i \in I. \tag{30}$$

The extension $\iota_\Gamma : \Gamma(B) \to \Gamma(\mathcal{P})$ of ι exists, fulfills

$$\chi_{i_\Gamma} \circ \iota_\Gamma(\gamma) = \pi_{i_\Gamma}(\gamma)\hat{\otimes}1 \quad \forall\gamma \in \Gamma(B),$$

and is injective.

Proof. As explained before Definition 4, the differential ideal corresponding to $\Gamma(\mathcal{P})$ is $J(\mathcal{P}) = \cap_i\ker\chi_{i_{\Omega\to\Gamma}} \subset \Omega(\mathcal{P})$. Using the right covariance of $\Omega(\mathcal{P})$ and $\Gamma(H)$, and Definition 15 one finds that the extensions $\chi_{i_{\Omega\to\Gamma}}$ fulfill

$$(\chi_{i_{\Omega\to\Gamma}} \otimes id) \circ \Delta^\Omega_{\mathcal{P}} = (id \otimes \Delta^\Gamma) \circ \chi_{i_{\Omega\to\Gamma}},$$

where Δ^Γ is the right coaction of $\Gamma(H)$. Due to this formula the differential ideals $\ker\chi_{i_{\Omega\to\Gamma}}$ are covariant under the coaction of H, i.e. $\Delta^\Omega_{\mathcal{P}}(\ker\chi_{i_{\Omega\to\Gamma}}) \subset \ker\chi_{i_{\Omega\to\Gamma}} \otimes H$, thus, the differential ideal $J(\mathcal{P}) := \cap_{i\in I}\ker\chi_{i_{\Omega\to\Gamma}}$ corresponding to $\Gamma(\mathcal{P})$ is covariant and it follows that $\Gamma(\mathcal{P})$ is covariant. This also gives (30).

The differential ideal corresponding to $\Gamma(B)$ is $J(B)=\cap_i \ker \pi_{i_{\Omega\to\Gamma}} \subset \Omega(B)$. It is easy to see that $\iota_\Omega(J(B)) \subset J(\mathcal{P})$, thus the extension ι_Γ of ι with respect to $\Gamma(B)$ and $\Gamma(\mathcal{P})$ exists. Clearly ι_Γ satisfies

$$\chi_{i_\Gamma} \circ \iota_\Gamma(\gamma) = \pi_{i_\Gamma}(\gamma)\hat{\otimes}1 \quad \forall\gamma \in \Gamma(B).$$

Because of this formula and $\cap_i \ker \pi_{i_\Gamma}=0$, ι_Γ is injective. □

The differential structure on a locally trivial QPFB determines the covering completion $\Gamma_c(\mathcal{P})$ of $\Gamma(\mathcal{P})$ with respect to the covering $(\ker \chi_{i_\Gamma})_{i\in I}$ (see Appendix A). $\Gamma_c(\mathcal{P})$ is an LC-differential algebra (see Appendix A) with local differential calculi $\Gamma(B_i)\hat{\otimes}\Gamma(H)$. It will be shown that $\Gamma_c(\mathcal{P})$ is a right H-comodule algebra and that the covering completion $\Gamma_c(B)$ of $\Gamma(B)$ is embedded in $\Gamma_c(\mathcal{P})$. But first we need some facts about differential calculi over $B_{ij}\otimes H$ appearing in our context. For the moment we can even assume that the above construction of $\Gamma(\mathcal{P})$ is performed for a general differential calculus $\Gamma(B_i\otimes H)$. Over the algebras $B_{ij}\otimes H$ there exist two isomorphic differential calculi $\Gamma^i(B_{ij}\otimes H)=\Gamma(B_i\otimes H)/\chi_{i_\Gamma}(\ker \chi_{j_\Gamma})$ and $\Gamma^j(B_{ij}\otimes H)=\Gamma(B_j\otimes H)/\chi_{j_\Gamma}(\ker \chi_{i_\Gamma})$, and two corresponding differential ideals $J^i(B_{ij}\otimes H)\subset \Omega(B_{ij}\otimes H)$ and $J^j(B_{ij}\otimes H)\subset \Omega(B_{ij}\otimes H)$.

Proposition 8. *The differential ideals $J^i(B_{ij}\otimes H)$ and $J^j(B_{ij}\otimes H)$ have the following form:*

$$J^i(B_{ij}\otimes H) = (\pi^i_j\otimes id)_\Omega(J(B_i\otimes H)) + \phi_{ij_\Omega}\circ(\pi^j_i\otimes id)_\Omega(J(B_j\otimes H)) \tag{31}$$

$$J^j(B_{ij}\otimes H) = (\pi^j_i\otimes id)_\Omega(J(B_j\otimes H)) + \phi_{ji_\Omega}\circ(\pi^i_j\otimes id)_\Omega(J(B_i\otimes H)), \tag{32}$$

where ϕ_{ij_Ω} are the the extensions of the isomorphisms ϕ_{ij} corresponding to the transition functions τ_{ji}.

For the proof, we need the following lemma.

Lemma 2.

$$(\pi^i_j\otimes id)\circ\chi_i = \phi_{ij}\circ(\pi^j_i\otimes id)\circ\chi_j.$$

Proof of the lemma. Using the identities $\phi_{ij}=\chi^i_{ij}\circ\chi^{j\,-1}_{ij}$ and $\chi^i_{ij}\circ\pi_{ij_\mathcal{P}}=(\pi^i_j\otimes id)\circ\chi_i$, one has

$$(\pi^i_j\otimes id)\circ\chi_i = \chi^i_{ij}\circ\pi_{ij_\mathcal{P}} = \phi_{ij}\circ\chi^j_{ij}\circ\pi_{ij_\mathcal{P}} = \phi_{ij}\circ(\pi^j_i\otimes id)\circ\chi_j. \qquad \square$$

Proof of the proposition. The differential calculus $\Gamma^i(B_{ij}\otimes H)=\Gamma(B_i\otimes H)/\chi_{i_\Gamma}(\ker \chi_{j_\Gamma})$ is isomorphic to $\Gamma(\mathcal{P})/(\ker \chi_{i_\Gamma} + \ker \chi_{j_\Gamma})$, which in turn is isomorphic to $\Omega(\mathcal{P})/(\ker \chi_{i_{\Omega\to\Gamma}} + \ker \chi_{j_{\Omega\to\Gamma}})$. Thus, the differential calculi $\Gamma^i(B_{ij}\otimes H)$ and $\Gamma^j(B_{ij}\otimes H)$ can be identified with $\Omega(B_i\otimes H)/\chi_{i_\Omega}(\ker \chi_{i_{\Omega\to\Gamma}} + \ker \chi_{j_{\Omega\to\Gamma}})$ and $\Omega(B_j\otimes H)/\chi_{j_\Omega}(\ker \chi_{i_{\Omega\to\Gamma}} + \ker \chi_{j_{\Omega\to\Gamma}})$, respectively. Applying $(\pi^i_j\otimes id)_\Omega$ respectively $(\pi^j_i\otimes id)_\Omega$ one obtains the differential ideals

$$J^i(B_{ij}\otimes H) = (\pi^i_j\otimes id)_\Omega\circ\chi_{i_\Omega}(\ker \chi_{i_{\Omega\to\Gamma}} + \ker \chi_{j_{\Omega\to\Gamma}}),$$

$$J^j(B_{ij}\otimes H) = (\pi^j_i\otimes id)_\Omega\circ\chi_{j_\Omega}(\ker \chi_{i_{\Omega\to\Gamma}} + \ker \chi_{j_{\Omega\to\Gamma}}).$$

Now, $\chi_{i_\Omega}(\ker\chi_{i_{\Omega\to\Gamma}}) = J(B_i\otimes H)$ and $\chi_{j_\Omega}(\ker\chi_{j_{\Omega\to\Gamma}}) = J(B_j\otimes H)$ yields

$$J^i(B_{ij}\otimes H) = (\pi^i_j\otimes id)_\Omega(J(B_i\otimes H)) + (\pi^i_j\otimes id)_\Omega\circ\chi_{i_\Omega}(\chi^{-1}_{j_\Omega}(J(B_j\otimes H))). \tag{33}$$

Due to Lemma 2, the two homomorphisms $(\pi^i_j\otimes id)_\Omega\circ\chi_{i_\Omega} : \Omega(\mathcal{P})\to\Omega(B_{ij}\otimes H)$ and $(\pi^j_i\otimes id)_\Omega\circ\chi_{j_\Omega} : \Omega(\mathcal{P})\to\Omega(B_{ij}\otimes H)$ are connected by

$$(\pi^i_j\otimes id)_\Omega\circ\chi_{i_\Omega} = \phi_{ij_\Omega}\circ(\pi^j_i\otimes id)_\Omega\circ\chi_{j_\Omega},$$

thus

$$(\pi^i_j\otimes id)_\Omega\circ\chi_{i_\Omega}(\chi^{-1}_{j_\Omega}(J(B_j\otimes H))) = \phi_{ij_\Omega}\circ(\pi^j_i\otimes id)_\Omega(J(B_j\otimes H)).$$

Inserting this formula in (33) gives (31). (32) results by exchanging i, j. □

Due to $J^i(B_{ij}\otimes H) = \phi_{ij_\Omega}(J^j(B_{ij}\otimes H))$ (immediate from Proposition 8) the isomorphism ϕ_{ij} is differentiable with respect to $\Gamma^j(B_{ij}\otimes H)$ and $\Gamma^i(B_{ij}\otimes H)$. ϕ_{ij_Γ} denotes the corresponding extension.

From now on, coming back to the description of $\Gamma_c(\mathcal{P})$, we consider the case $\Gamma(B_i\otimes H) = \Gamma(B_i)\hat{\otimes}\Gamma(H)$.

Denoting by $(\pi^i_j\otimes id)_{\Gamma^i} : \Gamma(B_i)\hat{\otimes}\Gamma(H)\to\Gamma^i(B_{ij}\otimes H)$ the natural projection, $\Gamma_c(\mathcal{P})$ has the following explicit form

$$\Gamma_c(\mathcal{P}) = \left\{(\gamma_i)_{i\in I}\in\bigoplus_{i\in I}\Gamma(B_i)\hat{\otimes}\Gamma(H)\,\middle|\,(\pi^i_j\otimes id)_{\Gamma^i}(\gamma_i)=\phi_{ij_\Gamma}\circ(\pi^j_i\otimes id)_{\Gamma^j}(\gamma_j)\right\}. \tag{34}$$

Note that this gluing is fully analogous to the gluing (16).

Proposition 9. *Let $\Gamma(\mathcal{P})$ be a differential structure on $\mathcal{P}$, let $\Gamma_c(\mathcal{P})$ be the covering completion of $\Gamma(\mathcal{P})$ and let $\Gamma_c(B)$ be the covering completion of $\Gamma(B)$. Let $\chi_{i_{\Gamma_c}}$ and $\pi_{i_{\Gamma_c}}$ be the restrictions of the respective ith projections.*

Then there exist a unique right coaction $\Delta^{\Gamma_c}_{\mathcal{P}} : \Gamma_c(\mathcal{P})\to\Gamma_c(\mathcal{P})\otimes H$ and a unique injective homomorphism $\iota_{\Gamma_c} : \Gamma_c(B)\to\Gamma_c(\mathcal{P})$ such that

$$(\chi_{i_{\Gamma_c}}\otimes id)\circ\Delta^{\Gamma_c}_{\mathcal{P}} = (id\otimes\Delta^\Gamma)\circ\chi_{i_{\Gamma_c}}, \tag{35}$$

$$\chi_{i_{\Gamma_c}}\circ\iota_{\Gamma_c}(\gamma) = \pi_{i_{\Gamma_c}}(\gamma)\otimes 1\quad\forall\gamma\in\Gamma_c(B). \tag{36}$$

Remark. Indeed, the $\chi_{i_{\Gamma_c}} : \Gamma_c(\mathcal{P})\to\Gamma(B_i)\hat{\otimes}\Gamma(H)$ and $\pi_{i_{\Gamma_c}} : \Gamma_c(B)\to\Gamma(B_i)$ coincide with differential extensions of χ_i and π_i, respectively. In the following proof, we will need

$$\ker(\pi^i_j\otimes id)_{\Gamma^i} = \chi_{i_\Gamma}(\ker\chi_{j_\Gamma}) = \chi_{i_{\Gamma_c}}(\ker\chi_{j_{\Gamma_c}}). \tag{37}$$

Proof. The ideals $\chi_{i_\Gamma}(\ker\chi_{j_\Gamma})$ are covariant under the H-coaction $(id_i\otimes\Delta^\Gamma)$, as follows from the covariance of the ideals $\ker\chi_{i_\Gamma}$ under the H-coaction $\Delta^\Gamma_{\mathcal{P}}$. Therefore, making use of the first equality (37), there exist H-coactions $(id\otimes\Delta)^{\Gamma^i}$ on $\Gamma^i(B_{ij}\otimes H)$ satisfying

$$(id\otimes\Delta)^{\Gamma^i}\circ(\pi^i_j\otimes id)_{\Gamma^i}=((\pi^i_j\otimes id)_{\Gamma^i})\otimes id)\circ(id\otimes\Delta^{\Gamma}), \tag{38}$$

$$(id\otimes\Delta)^{\Gamma^i}\circ\phi_{ij_\Gamma}=(\phi_{ij_\Gamma}\otimes id)\circ(id\otimes\Delta)^{\Gamma^j}. \tag{39}$$

Thus there exists an H-coaction $\Delta^{\Gamma_c}_{\mathcal{P}}$ on $\Gamma_c(\mathcal{P})$ defined by

$$\Delta^{\Gamma_c}_{\mathcal{P}}((\gamma_i)_{i\in I})=((id_i\otimes\Delta^{\Gamma})(\gamma_i))_{i\in I}\quad\forall(\gamma_i)_{i\in I}\in\Gamma_c(\mathcal{P}). \tag{40}$$

Further, one defines an injective homomorphism $\iota_{\Gamma_c}:\Gamma_c(B)\to\Gamma_c(\mathcal{P})$ by

$$\iota_{\Gamma_c}((\rho_i)_{i\in I})=(\rho_i\hat{\otimes}1)_{i\in I}\quad\forall(\rho_i)_{i\in I}\in\Gamma_c(B). \tag{41}$$

Both homomorphisms are uniquely determined by the assumptions of the proposition. □

In general, the differential calculi $\Gamma^i(B_{ij}\otimes H)$ and $\Gamma^j(B_{ij}\otimes H)$ seem not to be isomorphic to differential calculi of the form $\Gamma(B_{ij})\hat{\otimes}\Gamma(H)$. This is suggested by a look at the generators of the differential ideal $J^i(B_{ij}\otimes H)$.

Let $\iota_{i_\Omega}:\Omega(B_i)\to\Omega(B_i\otimes H)$ be the extension of $\iota_i:=id\otimes 1$ and let $\phi_{i_\Omega}:\Omega(H)\to\Omega(B_i\otimes H)$ be the extension of $\phi_i:=1\otimes id$. By Proposition 6, the differential ideal $J(B_i\otimes H)$ corresponding to $\Gamma(B_i)\hat{\otimes}\Gamma(H)$ is generated by the sets

$$\iota_{i_\Omega}(J(B_i)),\qquad\phi_{i_\Omega}(J(H)),$$

$$\{(a\otimes 1)d(1\otimes h)-(d(1\otimes h))(a\otimes 1),\quad a\in B_i,\ \ h\in H\}, \tag{42}$$

where the differential ideals $J(B_i)$ and $J(H)$ correspond to the differential calculi $\Gamma(B_i)$ and $\Gamma(H)$. Assume that the differential ideal $J(H)$ is determined by a right ideal $R\subset\ker\varepsilon\subset H$ in the sense that $J(H)$ is generated by the set $\{\sum S^{-1}(r_{(2)})dr_{(1)}|r\in R\}$ (see also Appendix A). Using (15), (42) and (31), one obtains the following generators of $J^i(B_{ij}\otimes H)$:

$$(\pi^i_j\otimes id)_\Omega\circ\iota_{i_\Omega}(J(B_i)),\ (\pi^j_i\otimes id)_\Omega\circ\iota_{j_\Omega}(J(B_j)), \tag{43}$$

$$\left\{\sum(1\otimes S^{-1}(r_{(2)}))d(1\otimes r_{(1)})\Big|\,r\in R\right\}, \tag{44}$$

$$\left\{\sum(\tau_{ij}(r_{(4)})\otimes S^{-1}(r_{(3)}))d(\tau_{ji}(r_{(1)})\otimes r_{(2)})\Big|\,r\in R\right\}, \tag{45}$$

$$\{(a\otimes 1)d(1\otimes h)-(d(1\otimes h))(a\otimes 1)|a\in B_{ij},h\in H\}, \tag{46}$$

$$\begin{aligned}\Big\{&(a\otimes 1)d\left(\sum\tau_{ji}(h_{(1)})\otimes h_{(2)}\right)\\&-\left(d\left(\sum\tau_{ji}(h_{(1)})\otimes h_{(2)}\right)\right)(a\otimes 1)\Big|\,a\in B_{ij},\ \ h\in H\Big\}.\end{aligned} \tag{47}$$

Observe that by (45)–(47) and the Leibniz rule

$$\begin{aligned}&\sum(\tau_{ij}(r_{(4)})\otimes S^{-1}(r_{(3)}))d(\tau_{ji}(r_{(1)}\otimes r_{(2)}))\\&\quad-\sum(\tau_{ij}(r_{(2)})\otimes 1)d(\tau_{ji}(r_{(1)})\otimes 1)\\&\quad-\sum(\tau_{ji}(S(r_4)r_{(1)})\otimes S^{-1}(r_{(3)}))d(1\otimes r_{(2)})\in J^i(B_{ij}\otimes H),\quad r\in R,\end{aligned}$$

thus one can replace the generators (45) by

$$\begin{aligned}&\sum(\tau_{ij}(r_{(2)})\otimes 1)d(\tau_{ji}(r_{(1)})\otimes 1)\\&\quad+\sum(\tau_{ji}(S(r_{(4)})r_{(1)})\otimes S^{-1}(r_{(3)}))d(1\otimes r_2)\in J^i(B_{ij}\otimes H),\quad r\in R.\end{aligned}\tag{48}$$

Using the Leibniz rule, the fact that the image of τ_{ji} lies in the centre of B_{ij}, and the generators (46), one can replace (47) by the set of generators

$$\{(a\otimes 1)d(\tau_{ji}(h)\otimes 1)-d(\tau_{ji}(h)\otimes 1)(a\otimes 1)|a\in B_{ij},\ \ h\in H\}.$$

Proposition 10. *Let the differential calculus $\Gamma(H)$ be determined by a right ideal $R\subset\ker\varepsilon\subset H$ and let τ_{ji} be the transition function corresponding to the isomorphism ϕ_{ij}. Assume that the right ideal has the property*

$$\sum\tau_{ij}(S(r_{(1)})r_{(3)})\otimes r_{(2)}\in B_{ij}\otimes R\quad\forall r\in R,\quad\forall i,j\in I.\tag{49}$$

Then there exist differential ideals $J_m(B_{ij})\subset\Omega(B_{ij})$ such that

$$\Gamma^i(B_{ij}\otimes H)=\Gamma^j(B_{ij}\otimes H)\cong(\Omega(B_{ij})/J_m(B_{ij}))\hat{\otimes}\Gamma(H).$$

Proof. Because of (49) the second term of (48) lies already in the part of $J^i(B_{ij}\otimes H)$ generated by the set (44), thus $J^i(B_{ij}\otimes H)$ is generated by the sets

$$\begin{aligned}&(\pi^i_j\otimes id)_\Omega\circ\iota_{i\Omega}(J(B_i)),\ (\pi^j_i\otimes id)_\Omega\circ\iota_{j\Omega}(J(B_j)),\\&\left\{\sum(1\otimes S^{-1}(r_{(2)}))d(1\otimes r_{(1)})\middle| r\in R\right\},\\&\left\{\sum(\tau_{ij}(r_{(2)})\otimes 1)d(\tau_{ji}(r_{(1)})\otimes 1)\middle| r\in R\right\},\\&\{(a\otimes 1)d(1\otimes h)-(d(1\otimes h))(a\otimes 1)|a\in B_{ij},h\in H\},\\&\{(a\otimes 1)d(\tau_{ji}(h)\otimes 1)-(d(\tau_{ji}(h)\otimes 1))(a\otimes 1)|a\in B_{ij},h\in H\}.\end{aligned}$$

One can see that the differential ideal $J^i(B_{ij}\otimes H)$ is of the form (27), where the differential ideal $J_m(B_{ij})$ corresponding to $\Omega(B_{ij})/J_m(B_{ij})$ is generated by the following sets:

$$\pi^i_{j_\Omega}(J(B_i)),\qquad\pi^j_{i_\Omega}(J(B_j)),\tag{50}$$

$$\left\{\sum\tau_{ji}(r_{(1)})d\tau_{ij}(r_{(2)})\middle| r\in R\right\},\tag{51}$$

$$\{(d\tau_{ji}(h))a-ad\tau_{ji}(h)|a\in B_{ij},h\in H\}.\tag{52}$$

Replacing i and j, we get the same differential ideal $J_m(B_{ij})$: the relation $\tau_{ji}(S(h)) = \tau_{ij}(h)$ gives invariance of the set (52) under this replacement. To see invariance of the set (51), we start from the identity

$$\sum \tau_{ij}(r_{(1)})\tau_{ji}(r_{(2)})d(\tau_{ij}(r_{(3)})\tau_{ji}(r_{(4)})) = 0, \quad r \in R$$

and obtain

$$\sum \tau_{ji}(S(r_{(1)})r_{(4)})\tau_{ji}(r_{(2)})d\tau_{ij}(r_{(3)}) + \sum \tau_{ij}(r_{(1)})d(\tau_{ji}(r_{(2)}) \in J_m(B_{ij}).$$

Due to (49) and (51) the first term lies already in $J_m(B_{ij})$, thus $\{\sum \tau_{ij}(r_{(1)})d(\tau_{ji}(r_{(2)})|r \in R\} \subset J_m(B_{ij})$. This shows that also $\Gamma^j(B_{ij} \otimes H) \simeq \Omega(B_{ij}/J_m(B_{ij})\hat{\otimes}\Gamma(H)$. □

Remark. All right ideals R determining a bicovariant differential calculus $\Gamma(H)$ have the property (49), because such right ideals are Ad-invariant, i.e. $\sum S(r_{(1)})r_{(3)} \otimes r_{(2)} \in H \otimes R \quad \forall r \in R$.

Observe that in the case described in the previous proposition the differential ideal $J_m(B_{ij})$ is in general larger than the differential ideal $J(B_{ij})$ (compare (50) with formula (14) of [6]), thus the differential calculi $\Gamma_m(B_{ij}) := \Omega(B_{ij})/J_m(B_{ij})$ and $\Gamma(B)/(\ker \pi_{i_\Gamma} + \ker \pi_{j_\Gamma})$ are in general not isomorphic.

Thus, we can define the differential algebra

$$\Gamma_m(B) := \left\{ (\gamma_i)_{i\in I} \in \bigoplus_i \Gamma(B_i) \middle| \pi^i_{j_{\Gamma_m}}(\gamma_i) = \pi^j_{i_{\Gamma_m}}(\gamma_j) \right\},$$

where the homomorphisms $\pi^j_{i_{\Gamma_m}} : \Gamma(B_i) \to \Gamma_m(B_{ij})$ are the compositions of the maps $\Gamma(B_{ij}) \to \Gamma_m(B_{ij})$ induced by the embedding $J(B_{ij}) \subset J_m(B_{ij})$ and $\pi^i_{j_\Gamma}$. Because of $J(B_{ij}) \subset J_m(B_{ij})$ the LC-differential algebra $\Gamma_c(B)$ is a subalgebra of $\Gamma_m(B)$. Further, $\Gamma_m(B)$ is an LC-differential algebra naturally embedded in $\Gamma_c(\mathcal{P})$ by $(\gamma_i)_{i\in I} \to (\gamma_i \otimes 1)_{i\in I}$. If (49) is fulfilled one has the identity

$$(\pi^i_j \otimes id)_{\Gamma^i} = \pi^i_{j_{\Gamma_m}} \otimes id.$$

If the right ideal R determining $\Gamma(H)$ does not fulfill (49), one can nevertheless construct an LC-differential algebra $\Gamma_m(B)$ with $\Gamma_c(B)$ as subalgebra, and this LC-differential algebra on B will naturally appear in the theory of connections on $\mathcal{P}$. For the definition of this LC-differential algebra, we need the following remark about the differential calculus induced on a subalgebra. Let C be an algebra and let $A \subset C$ be a subalgebra. From a differential calculus $\Gamma(C)$, one obtains a differential calculus $\Gamma(A)$ by

$$\Gamma^n(A) := \left\{ \sum_k a_0^k da_1^k \cdots da_n^k \in \Gamma(C) \middle| a_i^k \in A \right\}.$$

Let $J(C) \subset \Omega(C)$ be the differential ideal corresponding to the differential calculus $\Gamma(C)$. It is easy to verify that the differential ideal $J(A) \subset \Omega(A)$ corresponding to $\Gamma(A)$ is $J(C) \cap \Omega(A)$.

Now recall that there are differential calculi $\Gamma^i(B_{ij}\otimes H)$ and $\Gamma^j(B_{ij}\otimes H)$. Since $B_{ij}\otimes 1$ is a subalgebra of $B_{ij}\otimes H$, we obtain differential calculi $\Gamma^i(B_{ij})$ and $\Gamma^j(B_{ij})$ with corresponding differential ideals $J^i(B_{ij})$ and $J^j(B_{ij})$ defined by

$$J^i(B_{ij})\otimes 1 = J^i(B_{ij}\otimes H)\cap \Omega(B_{ij}\otimes 1),$$
$$J^j(B_{ij})\otimes 1 = J^j(B_{ij}\otimes H)\cap \Omega(B_{ij}\otimes 1).$$

From $\phi_{ij\Omega}(J^j(B_{ij}\otimes H)) = J^i(B_{ij}\otimes H)$ one concludes the identity $\phi_{ij\Omega}(J^j(B_{ij})) = J^i(B_{ij})$, and because of $\phi_{ij}(a\otimes 1) = a\otimes 1$ it follows that $J^i(B_{ij}) = J^j(B_{ij})$. Thus, we can define $\Gamma_m(B_{ij}) := \Gamma^i(B_{ij}) = \Gamma^j(B_{ij})$. There are injective homomorphisms $\iota^i_{ij\Gamma_m} : \Gamma_m(B_{ij}) \to \Gamma^i(B_{ij}\otimes H)$ given by

$$\iota^i_{ij\Gamma_m}(a_0 da_1 da_2\cdots da_n) = (a_0\otimes 1)d(a_1\otimes 1)d(a_2\otimes 1)\cdots d(a_n\otimes 1), \tag{53}$$

which fulfill the idenitity

$$\iota^i_{ij\Gamma_m} = \phi_{ij\Gamma}\circ \iota^j_{ij\Gamma_m}. \tag{54}$$

Let us define the projections $\pi^i_{j\Gamma_m} : \Gamma(B_i)\to \Gamma_m(B_{ij})$ and $\pi^j_{i\Gamma_m} : \Gamma(B_j)\to \Gamma_m(B_{ij})$ by

$$\iota^i_{ij\Gamma_m}\circ \pi^i_{j\Gamma_m}(\gamma_i) = (\pi^i_j\otimes id)_{\Gamma^i}(\gamma_i\hat{\otimes}1), \quad \gamma_i\in\Gamma(B_i), \tag{55}$$

$$\iota^j_{ij\Gamma_m}\circ \pi^j_{i\Gamma_m}(\gamma_j) = (\pi^j_i\otimes id)_{\Gamma^j}(\gamma_j\hat{\otimes}1), \quad \gamma_j\in\Gamma(B_j). \tag{56}$$

Obviously, these projections are extensions of π^i_j and π^j_i, respectively. In terms of these projections the LC-differential algebra $\Gamma_m(B)$ is defined as

$$\Gamma_m(B) := \left\{ (\gamma_i)_{i\in I}\in \bigoplus_{i\in I}\Gamma(B_i) \,\middle|\, \pi^i_{j\Gamma_m}(\gamma_i) = \pi^j_{i\Gamma_m}(\gamma_j)\right\}. \tag{57}$$

$\Gamma_{\mathrm{c}}(B)$ is a subalgebra of $\Gamma_m(B)$, and there exists an injective homomorphism $\iota_{\Gamma_m} : \Gamma_m(B)\to \Gamma_{\mathrm{c}}(\mathcal{P})$ defined by

$$\iota_{\Gamma_m}((\gamma_i)_{i\in I}) = (\gamma_i\hat{\otimes}1)_{i\in I}.$$

Example. We consider a $U(1)$ bundle over the sphere S^2. Assume that the algebra of differentiable functions $C^\infty(U(1))$ over $U(1)$ is the closure in some Fréchet topology of the algebra generated by the elements u and u^* satisfying

$$uu^* = u^*u = 1.$$

With $\Delta(u) = u\otimes u$, $\varepsilon(u) = 1$ and $S(u) = u^*$, this is a Hopf algebra. Let U_{N} and U_{S} be the (closed) northern and the southern hemisphere, respectively, $\{U_{\mathrm{N}}, U_{\mathrm{S}}\}$ is a covering of S^2. We have a complete covering $\{I_{\mathrm{N}}, I_{\mathrm{S}}\}$ of $C^\infty(S^2)$, $I_{\mathrm{N}}\subset C^\infty(S^2)$ and $I_{\mathrm{S}}\subset C^\infty(S^2)$ being the functions vanishing on the subsets U_{N} and U_{S}, respectively. Elements of $C^\infty(U_{\mathrm{N}}) = C^\infty(S^2)/I_{\mathrm{N}}$ and $C^\infty(U_{\mathrm{S}}) = C^\infty(S^2)/I_{\mathrm{S}}$ can be identified with restrictions of elements of

$C^\infty(S^2)$ to the subsets U_N and U_S, respectively. Since $U_\mathrm{N} \cap U_\mathrm{S} = S^1$, a transition function $\tau_\mathrm{NS} : C^\infty(U(1)) \to C^\infty(S^1)$ defines a locally trivial QPFB $\mathcal{P}$. We choose

$$\tau_\mathrm{NS}(u)(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{\mathrm{i}\phi}, \qquad \tau_\mathrm{NS}(u^*)(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{e}^{-\mathrm{i}\phi}$$

(Hopf bundle).

Now, we construct a differential structure on this bundle by fixing the differential calculi $\Gamma(C^\infty(U_\mathrm{N}))$, $\Gamma(C^\infty(U_\mathrm{S}))$ and $\Gamma(C^\infty(U(1)))$. $\Gamma(C^\infty(U_\mathrm{N}))$ and $\Gamma(C^\infty(U_\mathrm{S}))$ are taken to be the usual exterior differential calculi where the corresponding differential ideals are generated by all elements of the form $adb - dba$. For the right-covariant differential calculus $\Gamma(C^\infty(U(1)))$, we assume a noncommutative form. We choose as the right ideal R determining $\Gamma(C^\infty(U(1)))$ the right ideal generated by the element

$$u + \nu u^* - (1+\nu)1,$$

where $0 < \nu \le 1$ (one obtains the usual exterior differential calculus for $\nu = 1$).

Now, we are interested in the LC-differential algebra $\Gamma_m(C^\infty(S^2))$ coming from this differential structure on $\mathcal{P}$ for $\nu < 1$.

It is easy to verify that the right ideal R has the property (49), thus the differential ideal $J_m(C^\infty(S^1))$ is generated by the sets (50)–(52). The sets of generators (50) and (52) give the usual exterior differential calculus on S^1, but the set of generators (51) leads to $d\phi = qd\phi$, i.e. $d\phi = 0$ for $q < 1$. One obtains for the LC-differential algebra $\Gamma_m(C^\infty(S^2))$

$$\Gamma_m^0(C^\infty(S^2)) = C^\infty(S^2), \quad \Gamma_m^n(C^\infty(S^2)) = \Gamma^n\left(C^\infty(U_\mathrm{N})) \oplus \Gamma^n(C^\infty(U_\mathrm{S})\right), \quad n > 0.$$

The foregoing considerations suggest the following definition.

Definition 5. Let $\Gamma(\mathcal{P})$ be a differential structure on the locally trivial QPFB $\mathcal{P}$. An LC-differential algebra $\Gamma_g(B)$ over B is called embeddable into $\Gamma_\mathrm{c}(\mathcal{P})$ if the local differential calculi of $\Gamma_g(B)$ are $\Gamma(B_i)$ and if there exists the extension $\iota_{\Gamma_g} : \Gamma_g(B) \to \Gamma_\mathrm{c}(\mathcal{P})$ of ι such that

$$\chi_{i_{\Gamma_\mathrm{c}}} \circ \iota_{\Gamma_g}(\gamma) = \pi_{i_{\Gamma_g}}(\gamma)\hat{\otimes}1 \quad \forall \gamma \in \Gamma_g(B) \tag{58}$$

($\pi_{i_{\Gamma_g}} : \Gamma_g(B) \to \Gamma(B_i)$ is the extension of π_i).

Remark. From $\bigcap_{i\in I} \ker \pi_{i_{\Gamma_g}} = \{0\}$ it follows immediately that ι_{Γ_g} is injective.

Proposition 11. *The LC-differential algebra $\Gamma_m(B)$ defined above is the maximal embeddable LC-differential algebra, i.e every embeddable LC-differential algebra $\Gamma_g(B)$ is embedded in $\Gamma_m(B)$ as a subalgebra of the direct sum of the $\Gamma(B_i)$ by $\gamma \to (\pi_{i_{\Gamma_g}}(\gamma))_{i\in I}$.*

Proof. Let $\Gamma_g(B)$ be an embeddable LC-differential algebra. It is clear from $\bigcap_{i\in I} \ker \pi_{i_{\Gamma_g}} = \{0\}$ that the mapping is injective. To show that its image is in $\Gamma_m(B)$ one has to prove that for $\gamma \in \Gamma_g(B)$

$$\pi^i_{j_{\Gamma_m}} \circ \pi_{i_{\Gamma_g}}(\gamma) = \pi^j_{i_{\Gamma_m}} \circ \pi_{j_{\Gamma_g}}(\gamma) \tag{59}$$

(see (57)). By (58) ι_{Γ_g} has the form

$$\iota_{\Gamma_g}(\gamma) = (\pi_{i_{\Gamma_g}}(\gamma)\hat{\otimes}1)_{i\in I} \quad \forall\gamma \in \Gamma_g(B).$$

By definition, the image of ι_{Γ_g} lies in $\Gamma_c(\mathcal{P})$, i.e.

$$(\pi^i_j \otimes id)_{\Gamma^i}(\pi_{i_{\Gamma_g}}(\gamma)\hat{\otimes}1) = \phi_{ij_\Gamma} \circ (\pi^j_i \otimes id)_{\Gamma^j}(\pi_{j_{\Gamma_g}}(\gamma)\hat{\otimes}1).$$

Using (54)–(56) one obtains (59). □

4. Covariant derivatives and connections on locally trivial QPFB

This is the central section of this paper. We start by defining covariant derivatives, which are more general objects than connections. Only the latter are adapted to the right "group action" on the bundle. One of our main concerns will be to reconstruct all objects from objects of the same type given locally in the trivializations. For this it is crucial to work always with the covering completion $\Gamma_c(\mathcal{P})$ of a differential structure as given by formula (34).

Definition 6. Let $\Gamma(\mathcal{P})$ be the differential structure on $\mathcal{P}$ and let $\Gamma_c(\mathcal{P})$ be the covering completion of $\Gamma(\mathcal{P})$. Let hor $\Gamma_c(\mathcal{P}) \subset \Gamma_c(\mathcal{P})$ be the subalgebra defined by

$$\text{hor}\,\Gamma_c(\mathcal{P}) := \{\gamma \in \Gamma_c(\mathcal{P}) | \chi_{i_{\Gamma_c}}(\gamma) \in \Gamma(B_i)\hat{\otimes}H \quad \forall i \in I\}. \tag{60}$$

A linear map $D_{l,r}$: hor $\Gamma_c(\mathcal{P}) \to$ hor $\Gamma_c(\mathcal{P})$ is called left (right) covariant derivative if it satisfies

$$D_{l,r}(\text{hor}\,\Gamma^n_c(\mathcal{P})) \subset \text{hor}\,\Gamma^{n+1}_c(\mathcal{P}), \tag{61}$$

$$D_{l,r}(1) = 0, \tag{62}$$

$$D_l(\iota_{\Gamma_c}(\gamma)\alpha) = (d(\iota_{\Gamma_c}\gamma))\alpha + (-1)^n\gamma D_l(\alpha), \quad \gamma \in \Gamma^n_c(B), \;\; \alpha \in \text{hor}\,\Gamma_c(\mathcal{P}), \tag{63}$$

$$D_r(\alpha\iota_{\Gamma_c}(\gamma)) = D_r(\alpha)\iota_{\Gamma_c}(\gamma) + (-1)^n\alpha(d\iota_{\Gamma_c}(\gamma)), \quad \gamma \in \Gamma_c(B), \;\; \alpha \in \text{hor}\,\Gamma^n_c(\mathcal{P}), \tag{64}$$

$$(D_{l,r} \otimes id) \circ \Delta^{\Gamma_c}_{\mathcal{P}} = \Delta^{\Gamma_c}_{\mathcal{P}} \circ D_{l,r}, \tag{65}$$

$$D_{l,r}(\ker \chi_{i_{\Gamma_c}}|_{\text{hor}\,\Gamma_c(\mathcal{P})}) \subset \ker \chi_{i_{\Gamma_c}}|_{\text{hor}\,\Gamma_c(\mathcal{P})} \quad \forall i \in I. \tag{66}$$

In this definition the lower indices l or r indicate the left or the right case. The simultaneous appearance of both l, r means that the corresponding condition is fulfilled for both the left and the right case. This convention will be used in the sequel permanently.

Remark. In the case of trivial bundles $B \otimes H$ with differential structure $\Gamma(B)\hat{\otimes}\Gamma(H)$, where hor$(\Gamma(B)\hat{\otimes}\Gamma(H)) = \Gamma(B)\hat{\otimes}H$, condition (66) is trivial. Conditions (63) respectively (64) have the form

$$D_l(\gamma\hat{\otimes}h) = d\gamma\hat{\otimes}h + (-1)^n(\gamma\hat{\otimes}1)D_l(1 \otimes h), \quad \gamma \in \Gamma^n(B), \tag{67}$$

$$D_r(\gamma\hat{\otimes}h) = D_r(1\otimes h)(\gamma\hat{\otimes}1) + d\gamma\hat{\otimes}h. \tag{68}$$

Eq. (65) becomes

$$(D_{l,r}\otimes id)\circ(id\otimes\Delta^{\Gamma}) = (id\otimes\Delta^{\Gamma})\circ D_{l,r}. \tag{69}$$

Proposition 12. *Left (right) covariant derivatives are in bijective correspondence to families of linear maps* $A_{l,r_i}: H\to\Gamma^1(B_i)$ *with the properties*

$$A_{l,r_i}(1) = 0, \tag{70}$$

$$\pi^i_{j_{\Gamma_m}}(A_{l,r_i}(h)) = \sum\tau_{ij}(h_{(1)})\pi^j_{i_{\Gamma_m}}(A_{l,r_j}(h_{(2)}))\tau_{ji}(h_{(3)}) + \sum\tau_{ij}(h_{(1)})d\tau_{ji}(h_{(2)}). \tag{71}$$

Remark. Note that (71) is a condition in $\Gamma_m(B_{ij})$ (see the considerations at the end of the foregoing section).

Proof. Because of (66) a given left-covariant derivative on hor $\Gamma_{\mathrm{c}}(\mathcal{P})$ determines a family of left-covariant derivatives $D_{l_i}: \Gamma(B_i)\hat{\otimes}H\to\Gamma(B_i)\hat{\otimes}H$ by

$$D_{l_i}\circ\chi_{i_{\Gamma_{\mathrm{c}}}} = \chi_{i_{\Gamma_{\mathrm{c}}}}\circ D_l. \tag{72}$$

It follows the identity $D_l((\gamma_i)_{i\in I}) = (D_{l_i}(\gamma_i))_{i\in I}$. Since $(D_{l_i}(\gamma_i))_{i\in I}\in\Gamma_{\mathrm{c}}(\mathcal{P})$, the D_{l_i} satisfy

$$(\pi^i_j\otimes id)_{\Gamma^i}\circ D_{l_i}(\gamma_i) = \phi_{ij_\Gamma}\circ(\pi^j_i\otimes id)_{\Gamma^j}\circ D_{l_j}(\gamma_j), \quad (\gamma_i)_{i\in I}\in \mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P}). \tag{73}$$

One obtains a family of linear maps $A_{l_i}: H\to\Gamma^1(B_i)$ by

$$A_{l_i}(h) := -(id\otimes\varepsilon)\circ D_{l_i}(1\otimes h). \tag{74}$$

Now let us notice that the restriction of the right coaction $\Delta^{\Gamma}_{\mathcal{P}} = (id\otimes\Delta)^{\Gamma}$ to the horizontal forms $\Gamma(B)\otimes H$ of a trivial bundle just coincides with $id\otimes\Delta$. Moreover, this map can be composed with $id\otimes\varepsilon\otimes id$, and it is immediate from $(\varepsilon\otimes id)\circ\Delta = id$ that we have

$$(id\otimes\varepsilon\otimes id)\circ\Delta^{\Gamma}_{\mathcal{P}}|_{\Gamma(B)\otimes H} = id. \tag{75}$$

Using this formula, (67) and (69) one obtains the identity

$$D_{l_i}(\gamma\hat{\otimes}h) = d\gamma\hat{\otimes}h + (-1)^{n+1}\sum\gamma A_{l_i}(h_{(1)})\hat{\otimes}h_{(2)}, \quad \gamma\in\Gamma^n(B_i),\ \ h\in H. \tag{76}$$

Because of (62), the A_{l_i} fulfill (70). To prove the property (71), we need the following lemma.

Lemma 3. *Let* B *be an algebra,* H *be a Hopf algebra,* $\Gamma(B)$ *be a differential calculus over* B *and* $\Gamma(H)$ *be a right-covariant differential calculus over* H*. Let* $D_l: \Gamma(B)\hat{\otimes}H\to$

$\Gamma(B)\hat{\otimes}H$ be a left-covariant derivative on the trivial bundle $B\otimes H$. Let $J\subset\Gamma(B)\hat{\otimes}\Gamma(H)$ be a differential ideal with the property $(id\otimes\Delta^{\Gamma})(J)\subset J\otimes H$. Then, one has

$$D_l(J\cap(\Gamma(B)\hat{\otimes}H))\subset J\cap(\Gamma(B)\hat{\otimes}H). \tag{77}$$

Proof of the lemma. By Lemma 1 there is an ideal $\tilde{J}\subset\Gamma(B)$ such that

$$J\cap(\Gamma(B)\hat{\otimes}H)=\tilde{J}\hat{\otimes}H.$$

$\tilde{J}$ is a differential ideal. Let $\sum_k\gamma_k\hat{\otimes}h_k\in\tilde{J}\hat{\otimes}H\subset J$. Since J is a differential ideal one obtains

$$\sum_k d\gamma_k\hat{\otimes}h_k+(-1)^n\sum_k\gamma_k\hat{\otimes}dh_k\in J,\quad \gamma_k\in\Gamma^n(B).$$

The second summand lies in $\tilde{J}\hat{\otimes}\Gamma^1(H)\subset J$ again because J is an ideal. It follows that $\sum_k d\gamma_k\hat{\otimes}h_k\in d\tilde{J}\hat{\otimes}H\subset J\cap(\Gamma(B)\hat{\otimes}H)$ and one obtains $d\tilde{J}\subset\tilde{J}$, thus $\tilde{J}$ is a differential ideal.

Applying D_l to $\sum_k\gamma_k\hat{\otimes}h_k\in\tilde{J}\hat{\otimes}H\subset J$ leads to

$$D_l\left(\sum_k\gamma_k\hat{\otimes}h_k\right)=\sum_k d\gamma_k\hat{\otimes}h_k+(-1)^{n+1}\sum_k\gamma_k D_l(1\otimes h_k),\quad \gamma_k\in\Gamma^n(B).$$

Since the image of D_l lies in $\Gamma(B)\hat{\otimes}H$, the right-hand side of this formula is an element of $\tilde{J}\hat{\otimes}H$. □

Since the $\ker(\pi^i_j\otimes id)_{\Gamma^i}\subset\Gamma(B_i)\hat{\otimes}\Gamma(H)$ are coinvariant differential ideals (with respect to the coaction $id_i\otimes\Delta^{\Gamma}$, see (38)), the foregoing lemma shows $D_{l_i}(\ker(\pi^i_j\otimes id)_{\Gamma^i}\cap(\Gamma(B_i)\hat{\otimes}H))\subset\ker(\pi^i_j\otimes id)_{\Gamma^i}\cap(\Gamma(B_i)\hat{\otimes}H)$. This allows to define linear maps $D^{ij}_{l_i}:\Gamma^i(B_{ij}\otimes H)\to\Gamma^i(B_{ij}\otimes H)$ by

$$D^{ij}_{l_i}\circ(\pi^i_j\otimes id)_{\Gamma^i}=(\pi^i_j\otimes id)_{\Gamma^i}\circ D_{l_i}. \tag{78}$$

Applying $(\pi^i_j\otimes id)_{\Gamma^i}$ to (76) for $\gamma\in\Gamma^0(B_i)$, $a=\pi^i_j(\gamma)$, one obtains

$$D^{ij}_{l_i}(a\otimes h)=(d(a\otimes 1))(1\otimes h)-(a\otimes 1)\sum(\pi^i_j\otimes id)_{\Gamma^i}(A_{l_i}(h_{(1)})\otimes 1)(1\otimes h_{(2)}). \tag{79}$$

Let $(\gamma_i)_{i\in I}\in\text{hor}\,\Gamma_c(\mathcal{P})$, in particular

$$(\pi^i_j\otimes id)_{\Gamma^i}(\gamma_i)=\phi_{ij\Gamma}\circ(\pi^j_i\otimes id)_{\Gamma^j}(\gamma_j). \tag{80}$$

Since $D_l((\gamma_i)_{i\in I})=(D_{l_i}(\gamma_i))_{i\in I}\in\text{hor}\,\Gamma_c(\mathcal{P})$ it follows from (80) and (78) that

$$D^{ij}_{l_i}\circ(\pi^i_j\otimes id)_{\Gamma^i}(\gamma_i)=\phi_{ij\Gamma}\circ D^{ij}_{l_j}\circ(\pi^j_i\otimes id)_{\Gamma^j}(\gamma_j). \tag{81}$$

Combining (80) and (81), one obtains

$$D^{ij}_{l_i}=\phi_{ij\Gamma}\circ D^{ij}_{l_j}\circ\phi_{ji\Gamma}. \tag{82}$$

Taking advantage of (15), (53)–(55), (79) and (82), one computes

$$\begin{aligned} D^{ij}_{l_i}(1\otimes h) &= -\sum \iota^i_{ij\Gamma_m}(\pi^i_{j\Gamma_m}(A_{l_i}(h_{(1)})))(1\otimes h_{(2)}) = \phi_{ij\Gamma}\circ D^{ij}_{l_j}\circ\phi_{ji\Gamma}(1\otimes h)\\ &= \phi_{ij\Gamma}\circ D^{ij}_{l_j}\left(\sum \tau_{ij}(h_{(1)})\otimes h_{(2)}\right)\\ &= \phi_{ij\Gamma}\left(\sum \iota^j_{ij\Gamma_m}(d\tau_{ij}(h_{(1)}))(1\otimes h_{(2)})\right.\\ &\quad \left. -\sum \iota^j_{ij\Gamma_m}(\tau_{ij}(h_{(1)})\pi^j_{i\Gamma_m}(A_{l_j}(h_{(2)}))(1\otimes h_{(3)})\right)\\ &= \sum \iota^i_{ij\Gamma_m}((d\tau_{ij}(h_{(1)}))\tau_{ji}(h_{(2)})(1\otimes h_{(3)})\\ &\quad -\sum \iota^i_{ij\Gamma_m}(\tau_{ij}(h_{(1)})(\pi^j_{i\Gamma_m}(A_{l_j}(h_{(2)}))\tau_{ji}(h_{(3)}))(1\otimes h_{(4)}). \end{aligned} \tag{83}$$

Applying the Leibniz rule to the first term of the last row and using $\sum \tau_{ij}(h_{(1)})\tau_{ji}(h_{(2)}) = \varepsilon(h)1$, one obtains the identity

$$\begin{aligned} &\sum \iota^i_{ij\Gamma_m}(\pi^i_{j\Gamma_m}(A_{l_i}(h_{(1)})))(1\otimes h_{(2)})\\ &\quad = \sum \iota^i_{ij\Gamma_m}(\tau_{ij}(h_{(1)})\pi^j_{i\Gamma_m}(A_{l_j}(h_{(2)}))\tau_{ji}(h_{(3)}))(1\otimes h_{(4)})\\ &\qquad + \sum \iota^j_{ij\Gamma_m}(\tau_{ij}(h_{(1)})d(\tau_{ji}(h_{(2)})))(1\otimes h_{(3)}). \end{aligned} \tag{84}$$

In order to arrive at (71), we need to kill the $1\otimes h$-factor. This is achieved by using a projection $P_{\text{inv}} : \Gamma^i(B_{ij}\otimes H) \to \{\gamma \in \Gamma^i(B_{ij}\otimes H) | (id\otimes\Delta)^{\Gamma^i}(\gamma) = \gamma\otimes 1\}$ onto the elements of $\Gamma^i(B_{ij}\otimes H)$ being coinvariant under the right H coaction $(id\otimes\Delta)^{\Gamma^i} : \Gamma^i(B_{ij}\otimes H) \to \Gamma^i(B_{ij}\otimes H)\otimes H$ (see also (38) and (39)). P_{inv} is defined by

$$P_{\text{inv}}(\rho) = \sum \rho_{(0)}S(\rho_{(1)}), \quad \rho\in\Gamma^i(B_{ij}\otimes H). \tag{85}$$

Note that $P_{\text{inv}}(\rho(1\otimes h)) = \varepsilon(h)P_{\text{inv}}(\rho)$. Applying P_{inv} to the identity (84) leads to

$$\begin{aligned} \iota^i_{ij\Gamma_m}(\pi^i_{j\Gamma_m}(A_{l_i}(h))) &= \sum \iota^i_{ij\Gamma_m}(\tau_{ij}(h_{(1)})\pi^j_{i\Gamma_m}(A_{l_j}(h_{(2)}))\tau_{ji}(h_{(3)}))\\ &\quad + \sum \iota^i_{ij\Gamma_m}(\tau_{ij}(h_{(1)})d\tau_{ji}(h_{(2)})). \end{aligned}$$

Due to the injectivity of $\iota^i_{ij\Gamma_m}$, this is identical to

$$\pi^i_{j\Gamma_m}(A_{l_i}(h)) = \sum \tau_{ij}(h_{(1)})\pi^j_{i\Gamma_m}(A_{l_j})(h_{(2)}))\tau_{ij}(h_{(3)}) + \sum \tau_{ij}(h_{(1)})d\tau_{ji}(h_{(2)}) \tag{86}$$

in $\Gamma_m(B_{ij})$.

Now, we prove conversely that every family of linear maps $A_{l_i} : H \to \Gamma^1(B_i)$ with the properties (70) and (71) defines a left-covariant derivative. Assume that there is given such a family $(A_{l_i})_i$. Every A_{l_i} defines by

$$D_{l_i}(\gamma\hat{\otimes}h) = d\gamma\hat{\otimes}h + (-1)^{n+1}\sum \gamma A_{l_i}(h_{(1)})\hat{\otimes}h_{(2)}, \quad \gamma\in\Gamma^n(B_i),\ \ h\in H$$

a left-covariant derivative D_{l_i} on $\Gamma(B_i)\hat{\otimes}H$. The properties (61)–(63) and (65) of D_{l_i}, are easily derived from the above formula. One has to show that $D_l((\gamma_i)_{i\in I}) := (D_{l_i}(\gamma_i))_{i\in I}$,

$(\gamma_i)_{i\in I} \in \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$ is a covariant derivative on $\mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$. Because of (70), D_l fulfills (62). The conditions (63) and (65) follows from the corresponding properties of D_{l_i}. It remains to prove that the image of D_l lies in $\Gamma_{\mathrm{c}}(\mathcal{P})$, because then it also lies in $\mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$. (This is due to the fact that all the images of the D_{l_i} obviously are in $\Gamma(B_i)\hat{\otimes}H$.) Then, it is also obvious from the fact that the $\chi_{i_{\Gamma_{\mathrm{c}}}}$ are the projections to the ith components that condition (66) is fulfilled. The image of D_l lies in $\mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$ if the family of the D_{l_i} fulfills

$$(\pi^i_j \otimes id)_{\Gamma^i} \circ D_{l_i}(\gamma_i) = \phi_{ij\Gamma} \circ (\pi^j_i \otimes id)_{\Gamma^j} \circ D_{l_j}(\gamma_j) \quad \forall (\gamma_i)_{i\in I} \in \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P}). \quad (87)$$

By Lemma 3, the covariant derivatives D_{l_i} give rise to maps $D^{ij}_{l_i}$ defined by

$$D^{ij}_{l_i} \circ (\pi^i_j \otimes id)_{\Gamma^i} = (\pi^i_j \otimes id)_{\Gamma^i} \circ D_{l_i}.$$

One has

$$D^{ij}_{l_i}(1\otimes h) = -\sum (\pi^i_j \otimes id)_{\Gamma^i}(A_{l_i}(h_{(1)})\otimes 1)(1\otimes h_{(2)}), \quad (88)$$

and we will show that (71) yields the identity

$$D^{ij}_{l_i} = \phi_{ij\Gamma} \circ D^{ij}_{l_j} \circ \phi_{ji\Gamma}.$$

One computes for $\gamma \in \Gamma^n_m(B_{ij})$

$$\begin{aligned} & D^{ij}_{l_i}(\iota^i_{ij\Gamma_m}(\gamma)(1\otimes h)) \\ &= (d\gamma)(1\otimes h) + (-1)^{n+1}\iota^i_{ij\Gamma_m}(\gamma) D^{ij}_{l_i}(1\otimes h) \\ &= \iota^i_{ij\Gamma_m}(d\gamma)(1\otimes h) + (-1)^{n+1}\sum \iota^i_{ij\Gamma_m}(\gamma\pi^i_{j\Gamma_m}(A_{l_i}(h_{(1)})))(1\otimes h_{(2)}) \\ &= \iota^i_{ij\Gamma_m}(d\gamma)(1\otimes h) + (-1)^{n+1}\sum \iota^i_{ij\Gamma_m}(\gamma\tau_{ij}(h_{(1)})\pi^j_{i\Gamma_m}(A_{l_j}(h_{(2)}))\tau_{ji}(h_{(3)}))(1\otimes h_{(4)}) \\ &\quad + (-1)^{n+1}\sum \iota^i_{ij\Gamma_m}(\gamma(d\tau_{ij}(h_{(1)}))\tau_{ji}(h_{(2)}))(1\otimes h_{(3)}) \\ &= \phi_{ij\Gamma} \circ D^{ij}_{l_j} \circ \phi_{ji\Gamma}(\gamma(1\otimes h)). \end{aligned}$$

Thus, one obtains for $(\gamma_i)_{i\in I} \in \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$

$$D^{ij}_{l_i} \circ (\pi^i_j \otimes id)_{\Gamma^i}(\gamma_i) = D^{ij}_{l_i} \circ \phi_{ij\Gamma} \circ (\pi^j_i \otimes id)_{\Gamma^j}(\gamma_j) = \phi_{ij\Gamma} \circ D^{ij}_{l_j} \circ (\pi^j_i \otimes id)_{\Gamma^j}(\gamma_j),$$

and (87) follows.

It is immediate from the construction (using (75)) that the correspondence is bijective.

The proof for right-covariant derivatives is analogous. In this case one uses

$$D_{r_i}(\gamma\hat{\otimes}h) = d\gamma\hat{\otimes}h + (-1)^{n+1}\sum A_{r_i}(h_{(1)})\gamma\hat{\otimes}h_{(2)} \quad (89)$$

for $\gamma \in \Gamma^n(B_i)$. □

Remark. Obviously, a family of linear maps $A_i : H \to \Gamma^1(B_i)$ fulfilling (70) and (71) determines at the same time a left and a right covariant derivative. Consequently, there is a bijective correspondence between left and right covariant derivatives.

Proposition 13. *Let $D_{l,r}$: hor $\Gamma_{\rm c}(\mathcal{P}) \to$ hor $\Gamma_{\rm c}(\mathcal{P})$ be a left (right) covariant derivative and let $\Gamma_g(B)$ be embeddable into $\Gamma_{\rm c}(\mathcal{P})$. $D_{l,r}$ fulfills*

$$D_l(\iota_{\Gamma_g}(\gamma)\alpha) = (d(\iota_{\Gamma_g}(\gamma))\alpha + (-1)^n \iota_{\Gamma_g}(\gamma) D_l(\alpha), \quad \gamma \in \Gamma_g^n(B),\ \ \alpha \in \text{hor}\, \Gamma_{\rm c}(\mathcal{P}), \tag{90}$$

$$D_r(\alpha\iota_{\Gamma_g}(\gamma)) = D_r(\alpha)\iota_{\Gamma_g}(\gamma) + (-1)^n \alpha(d\iota_{\Gamma_g}(\gamma)), \quad \gamma \in \Gamma_g(B),\ \ \alpha \in \text{hor}\, \Gamma_{\rm c}^n(\mathcal{P}). \tag{91}$$

Proof. Let $(\gamma_i)_{i\in I} \in$ hor $\Gamma_{\rm c}(\mathcal{P})$ and $\rho \in \Gamma_g^n(B)$. One has $\iota_{\Gamma_g}(\rho) = (\pi_{i_{\Gamma_g}}(\rho)\hat{\otimes}1)_{i\in I}$ and $D_l((\gamma_i)_{i\in I}) = (D_{l_i}(\gamma_i))_{i\in I}$. One calculates

$$\begin{aligned} D_l(\iota_{\Gamma_g}(\rho)(\gamma_i)_{i\in I}) &= D_l(((\pi_{i_{\Gamma_g}}(\rho)\hat{\otimes}1)\gamma_i)_{i\in I}) = (D_{l_i}(\pi_{i_{\Gamma_g}}(\rho)\hat{\otimes}1)\gamma_i))_{i\in I} \\ &= ((d(\pi_{i_{\Gamma_g}}(\rho)\hat{\otimes}1))\gamma_i))_{i\in I} + (-1)^n((\pi_{i_{\Gamma_g}}(\rho)\hat{\otimes}1)D_{l_i}(\gamma_i))_{i\in I} \\ &= (d(\iota_{\Gamma_g}(\rho)))(\gamma_i)_{i\in I} + (-1)^n \iota_{\Gamma_g}(\rho) D_l((\gamma_i)_{i\in I}). \end{aligned}$$

The proof for right covariant derivatives is analogous. □

Now, we are going to define connections on locally trivial QPFB. It turns out that connections are special cases of covariant derivatives. We start with a definition dualizing the classical one in a certain sense.

Definition 7. Let $\Gamma(\mathcal{P})$ be a differential structure on $\mathcal{P}$ and let $\Gamma_{\rm c}(\mathcal{P})$ be the covering completion of $\Gamma(\mathcal{P})$. A left (right) connection is a surjective left (right) $\mathcal{P}$-module homomorphism $\text{hor}_{l,r} : \Gamma_{\rm c}^1(\mathcal{P}) \to \text{hor}\, \Gamma_{\rm c}^1(\mathcal{P})$ such that

$$\text{hor}_{l,r}^2 = \text{hor}_{l,r}, \tag{92}$$

$$(\text{hor}_{l,r} \otimes id) \circ \Delta_{\mathcal{P}}^{\Gamma_{\rm c}} = \Delta_{\mathcal{P}}^{\Gamma_{\rm c}} \circ \text{hor}_{l,r} \tag{93}$$

and

$$\text{hor}_{l,r}(\ker \chi_{i_{\Gamma_{\rm c}}}) \subset \ker \chi_{i_{\Gamma_{\rm c}}} \quad \forall i \in I. \tag{94}$$

Remark. Conditions (94) in this definition are needed to have a one-to-one correspondence between connections on $\mathcal{P}$ and certain families of connections on the trivial bundles $B_i \otimes H$. On a trivial bundle $B \otimes H$ condition (94) is obsolete.

Remark. For a given left (right) connection there is a vertical left (right) $\mathcal{P}$-submodule $\text{ver}_{l,r}\Gamma_{\rm c}^1(\mathcal{P})$ such that

$$\Gamma_{\rm c}^1(\mathcal{P}) = \text{ver}_{l,r}\Gamma_{\rm c}^1(\mathcal{P}) \oplus \text{hor}\Gamma_{\rm c}^1(\mathcal{P}),$$

where the projection $\text{ver}_{l,r} : \Gamma_{\rm c}^1(\mathcal{P}) \to \text{ver}_{l,r}\Gamma_{\rm c}^1(\mathcal{P})$ is defined by $\text{ver}_{l,r} := id - \text{hor}_{l,r}$. Obviously, $\text{ver}_{l,r}$ also fulfills Eqs. (92)–(94) and, moreover, $\ker v_{l,r} = \text{hor}\Gamma_{\rm c}^1(\mathcal{P})$. Conversely, a module map $\text{ver}_{l,r}$ satisfying these conditions defines the connection $\text{hor}_{l,r} =$

$1-\mathrm{ver}_{l,r}$. Thus one can say that a connection is nothing else but the choice of a "vertical" complement to the canonically given submodule of horizontal forms.

On a trivial bundle $B\otimes H$ with differential structure $\Gamma(B)\hat{\otimes}\Gamma(H)$ exists always the canonical connection $\mathrm{hor}_{\mathrm{c}}$, which is at the same time left and right. The existence of $\mathrm{hor}_{\mathrm{c}}$ comes from the decomposition

$$(\Gamma(B)\hat{\otimes}\Gamma(H))^1=(\Gamma^1(B)\hat{\otimes}H)\oplus(B\hat{\otimes}\Gamma^1(H))$$

(direct sum of $(B\otimes H)$-bimodules), which allows to define $\mathrm{hor}_{\mathrm{c}}$ as projection to the first component,

$$\mathrm{hor}_{\mathrm{c}}(\gamma\hat{\otimes}h)=\gamma\hat{\otimes}h,\quad \gamma\in\Gamma^1(B),\ \ h\in H,$$

$$\mathrm{hor}_{\mathrm{c}}(a\hat{\otimes}\theta)=0,\quad a\in B,\ \ \theta\in\Gamma^1(H).$$

Lemma 4. *For a given connection* $\mathrm{hor}_{l,r}$ *on* $\mathcal{P}$ *there exists a family of connections* hor_{l,r_i} *on the trivial bundles* $B_i\otimes H$ *such that*

$$\chi_{i_{\Gamma_{\mathrm{c}}}}\circ\mathrm{hor}_{l,r}=\mathrm{hor}_{l,r_i}\circ\chi_{i_{\Gamma_{\mathrm{c}}}}. \tag{95}$$

Proof. The existence of linear map hor_{l_i} satisfying (95) follows from (94). The hor_{l,r_i} are connections on $B_i\otimes H$: because of the surjectivity of the $\chi_{i_{\Gamma_{\mathrm{c}}}}$ the hor_{l_i} map onto $\Gamma^1(B_i)\hat{\otimes}H$. To prove condition (92) one computes

$$\mathrm{hor}^2_{l,r_i}\circ\chi_{i_{\Gamma_{\mathrm{c}}}}=\mathrm{hor}_{l,r_i}\circ\chi_{i_{\Gamma_{\mathrm{c}}}}\circ\mathrm{hor}_{l,r}=\chi_{i_{\Gamma_{\mathrm{c}}}}\circ\mathrm{hor}^2_{l,r}=\chi_{i_{\Gamma_{\mathrm{c}}}}\circ\mathrm{hor}_{l,r}=\mathrm{hor}_{l,r_i}\circ\chi_{i_{\Gamma_{\mathrm{c}}}}.$$

The condition (93) is fulfilled because of (35). □

By Definition 7 and the foregoing lemma a connection $\mathrm{hor}_{l,r}$ has the following form:

$$\mathrm{hor}_{l,r}((\gamma_i)_{i\in I})=(\mathrm{hor}_{l,r_i}(\gamma_i))_{i\in I},\quad (\gamma_i)_{i\in I}\in\mathrm{hor}\Gamma_{\mathrm{c}}(\mathcal{P}), \tag{96}$$

which also means that the family of linear maps hor_{l,r_i} satisfies

$$(\pi^i_j\otimes id)_{\Gamma^i}\circ\mathrm{hor}_{l,r_i}(\gamma_i)=\phi_{ij_\Gamma}\circ(\pi^j_i\otimes id)_{\Gamma^j}\circ\mathrm{hor}_{l,r_j}(\gamma_j) \tag{97}$$

for $(\gamma_i)_{i\in I}\in\Gamma^1_{\mathrm{c}}(\mathcal{P})$.

Proposition 14. *Let $R\subset H$ be the right ideal corresponding to the right-covariant differential calculus $\Gamma(H)$. Left (right) connections on a locally trivial QPFB $\mathcal{P}$ are in one-to-one correspondence to left (right) covariant derivatives with the following property: The corresponding linear maps A_{l,r_i} fulfill*

$$R\subset\ker A_{l_i}\quad\forall i\in I, \tag{98}$$

$$S^{-1}(R)\subset\ker A_{r_i}\quad\forall i\in I. \tag{99}$$

Remark. Thus left (right) connections are in one-to-one correspondence to linear maps A_{l,r_i} fulfilling (70), (71) and (98) (respectively (99)).

Proof. We perform the proof only for left connections. It is fully analogous for right connections.

A left connection hor_l determines a family of linear maps $A_{l_i} : H \to \Gamma^1(B_i)$ by

$$A_{l_i}(h) := -(id \otimes \varepsilon)\,\mathrm{hor}_{l_i}(1\hat{\otimes} dh).$$

From (93) and (75) one concludes the identity

$$\mathrm{hor}_{l_i}(1 \otimes dh) = -\sum A_{l_i}(h_{(1)})\hat{\otimes} h_{(2)}. \tag{100}$$

Therefore, A_{l_i} have the property $R \subset \ker A_{l_i}$

$$0 = \sum \mathrm{hor}_{l_i}(1\hat{\otimes} S^{-1}(r_{(2)})dr_{(1)}) = -A_{l_i}(r)\hat{\otimes}1, \quad r \in R.$$

It remains to show that the A_{l_i} fulfill (70) and (71).

Eq. (70) is fulfilled by definition ($A_{l_i}(1) := (id \otimes \varepsilon) \circ \mathrm{hor}_{l_i}(1 \otimes d1) = 0$).

Because of (94), (95) and (37), one has $\mathrm{hor}_{l_i}(\ker(\pi^i_j \otimes id)_{\Gamma^i}) \subset \ker(\pi^i_j \otimes id)_{\Gamma^i}$, and the linear maps $\mathrm{hor}^{ij}_{l_i}$ defined by

$$\mathrm{hor}^{ij}_{l_i} \circ (\pi^i_j \otimes id)_{\Gamma^i} = (\pi^i_j \otimes id)_{\Gamma^i} \circ \mathrm{hor}_{l_i}$$

exist. It follows that

$$\mathrm{hor}^{ij}_{l_i}(d(1 \otimes h)) = -\sum (\pi^i_{j_{\Gamma_m}}(A_{l_i}(h_{(1)})))(1 \otimes h_{(2)}).$$

On the other hand, by an analogue of the computation leading to (82) (using (97)), one obtains

$$\mathrm{hor}^{ij}_{l_i} = \phi_{ij_\Gamma} \circ \mathrm{hor}^{ij}_{l_j} \circ \phi_{ji_\Gamma}.$$

Using now the last two formulas, on can repeat the arguments written after formula (82) to obtain formula (71).

Now assume that there is given a left-covariant derivative D_l, whose corresponding linear maps A_{l_i} satisfy $R \subset \ker A_{l_i}$. There exist left connections $\mathrm{hor}_{l_i} : (\Gamma^1(B_i)\hat{\otimes}H) \oplus (B_i\hat{\otimes}\Gamma^1(H)) \to \Gamma^1(B_i)\hat{\otimes}H$ defined by

$$\mathrm{hor}_{l_i}(\gamma\hat{\otimes}h) := \gamma\hat{\otimes}h, \qquad \mathrm{hor}_{l_i}(a\hat{\otimes}hdk) := -\sum aA_{l_i}(k_{(1)})\hat{\otimes}hk_{(2)}. \tag{101}$$

To verify this assertion, we define linear maps $\mathrm{hor}^{\Omega}_{l_i} : (\Gamma(B_i)\hat{\otimes}\Omega(H))^1 \to \Gamma^1(B_i)\hat{\otimes}H$ by

$$\mathrm{hor}^{\Omega}_{l_i}(a_0da_1\hat{\otimes}h) = a_0da_1\hat{\otimes}h, \qquad \mathrm{hor}^{\Omega}_{l_i}(a\hat{\otimes}h^0dk) = -\sum aA_l(k_{(1)})\hat{\otimes}hk_{(2)}.$$

The $B_i \otimes H$-subbimodules $B_i\hat{\otimes}J^1(H)$ are generated by the sets $\{1\hat{\otimes}\sum S^{-1}(r_{(2)})dr_{(1)} | r \in R\}$. One has

$$B_i\hat{\otimes}\Gamma^1(H) = (B_i\hat{\otimes}\Omega^1(H))/(B_i\hat{\otimes}J^1(H)) = B_i\hat{\otimes}\Omega^1(H)/J^1(H).$$

Using $R \subset \ker A_{l_i}$ it is easy to verify that the linear maps $\mathrm{hor}_{l_i}^{\Omega}$ send $B_i \hat{\otimes} J^1(H)$ to zero, i.e. there exist corresponding linear maps hor_{l_i} on $(\Gamma(B_i)\hat{\otimes}\Gamma(H))^1$. As a consequence of their definition these linear maps are connections. One easily verifies the identity

$$\mathrm{hor}_{l_i} \circ d = D_{l_i}|_{B_i \otimes H}, \tag{102}$$

where the D_{l_i} are the local left-covariant derivatives defined by (72).

Now, we define a linear map $\mathrm{hor}_l : \Gamma_c^1(\mathcal{P}) \to \oplus_{i\in I}\Gamma(B_i)\hat{\otimes}H$ by

$$\mathrm{hor}_l((\gamma_i)_{i\in I}) := (\mathrm{hor}_{l_i}(\gamma_i))_{i\in I}, \quad (\gamma_i)_{i\in I} \in \Gamma_c^1(\mathcal{P}).$$

It remains to prove that the image of hor_l lies in $\Gamma_c^1(\mathcal{P})$. Then it will follow immediately from the properties of the local connections hor_{l_i} that hor_l is a connection.

To prove $\mathrm{hor}_l(\Gamma_c^1(\mathcal{P}) \subset \Gamma_c^1(\mathcal{P})$, we need a lemma.

Lemma 5. $\mathrm{hor}_{l_i}((\chi_{i\Gamma}(\ker\chi_{j\Gamma}))^1) \subset (\chi_{i\Gamma}(\ker\chi_{j\Gamma}))^1$

Proof of the lemma. Using the form (43)–(47) of the generators of $J^i(B_{ij}\otimes H)$ one finds easily that the differential calculus $\Gamma^i(B_{ij}\otimes H)$ has the form $\Gamma^i(B_{ij}\otimes H) = (\Gamma(B_{ij})\hat{\otimes}\Gamma(H))/J$, where the differential ideal J is generated by

$$\left\{\sum \tau_{ij}(r_{(2)})d\tau_{ji}(r_{(1)})\hat{\otimes}1 + \sum \tau_{ij}(r_{(4)})\tau_{ji}(r_{(1)})\hat{\otimes}S^{-1}(r_{(3)})dr_{(2)}\middle| r\in R\right\}, \tag{103}$$

$$\left\{\sum (ad\tau_{ji}(h) - (d\tau_{ji}(h))a)\hat{\otimes}1\middle| h\in H,\ a\in B_i\right\}. \tag{104}$$

These elements arise from (43)–(47) applying the map $id_{\Omega\to\Gamma} : \Omega(B_{ij}\otimes H) \to \Gamma(B_{ij})\hat{\otimes}\Gamma(H)$. Then, by definition, $J = (\pi^i_{j\Gamma}\otimes id)(\chi_{i\Gamma}(\ker\chi_{j\Gamma}))$.

The factorization map $id^i_{ij\Gamma} : \Gamma(B_{ij})\hat{\otimes}\Gamma(H)) \to \Gamma^i(B_{ij}\otimes H)$ fulfills

$$id^i_{ij\Gamma} \circ (\pi^i_{j\Gamma}\otimes id) = (\pi^i_j \otimes id)_\Gamma. \tag{105}$$

Since hor_{l_i} is a left module homomorphism and $\ker(\pi^i_{j\Gamma}\otimes id) = \ker\pi^i_{j\Gamma}\hat{\otimes}\Gamma(H)$, one has

$$\mathrm{hor}_{l_i}((\ker(\pi^i_{j\Gamma}\otimes id))^1) \subset (\ker(\pi^i_{j\Gamma}\otimes id))^1, \tag{106}$$

thus hor_{l_i} defines a connection $\mathrm{hor}^{ij}_{l_i} : (\Gamma(B_{ij})\hat{\otimes}\Gamma(H))^1 \to \Gamma^1(B_{ij})\hat{\otimes}H$ by

$$\mathrm{hor}^{ij}_{l_i} \circ (\pi^i_{j\Gamma}\otimes id) = (\pi^i_{j\Gamma}\otimes id)\circ \mathrm{hor}_{l_i}. \tag{107}$$

Because of (37) and (106) and

$$(\pi^i_j\otimes id)_\Gamma \circ \mathrm{hor}_{l_i}(\chi_{i\Gamma}(\ker\chi_{j\Gamma})) = id^i_{ij\Gamma}\circ \mathrm{hor}^{ij}_{l_i}(J)$$

which is immediate from (105) and (107), to prove the assertion of the lemma, we have to show that $id^i_{ij\Gamma}\circ\mathrm{hor}^{ij}_{l_i}(J) = 0$. Note that the part of J generated by (104) lies in the horizontal submodule $\Gamma^1(B_{ij})\hat{\otimes}H$ and is therefore invariant under $\mathrm{hor}^{ij}_{l_i}$. Concerning the part of J generated by (103), we argue as follows. Since $\mathrm{hor}^{ij}_{l_i}$ is a left module homomorphism, it is sufficient to consider the product of the generators (103) with a general element $(a \otimes h) \in B_{ij} \otimes H$ on the right. Using $R \subset \ker \varepsilon$, such an element can be written

$$\begin{aligned}&\sum \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}) a \hat{\otimes} h + \sum \tau_{ij}(r_{(4)}) \tau_{ji}(r_{(1)}) \hat{\otimes} S^{-1}(r_{(3)}) dr_{(2)})(a \otimes h)\\ &\quad = \sum \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}) a \hat{\otimes} h + \sum \tau_{ij}(r_{(4)}) \tau_{ji}(r_{(1)}) a \hat{\otimes} S^{-1}(r_{(3)}) d(r_{(2)} h),\\ &r \in R, \quad h \in H, \quad a \in B_{ij}.\end{aligned}$$

Using (101), $R \subset \ker \varepsilon$, (71), $R \subset \ker A_{l_j}$ and (53), one calculates

$$\begin{aligned}&id^{i}_{ij\Gamma} \circ \mathrm{hor}^{ij}_{l_i} \left(\sum \tau_{ij}(r_{(2)})(d\tau_{ji}(r_{(1)})) a \hat{\otimes} h + \sum \tau_{ij}(r_{(4)}) \tau_{ji}(r_{(1)}) a \hat{\otimes} S^{-1}(r_{(3)}) d(r_{(2)} h)\right)\\ &\quad = \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}))(1 \otimes h)\\ &\qquad - \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(3)}) \tau_{ji}(r_{(1)}) \pi^{i}_{j\Gamma_m}(A_{l_i}(r_{(2)} h_{(1)})))(1 \otimes h_{(2)})\\ &\quad = \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}))(1 \otimes h)\\ &\qquad - \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(5)}) \tau_{ji}(r_{(1)}) \tau_{ij}(r_{(2)} h_{(1)}) \tau_{ji}(r_{(4)} h_{(3)}) \pi^{j}_{i\Gamma_m}(A_{l_j}(r_{(3)} h_{(2)})))(1 \otimes h_{(4)})\\ &\qquad - \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(4)}) \tau_{ji}(r_{(1)}) \tau_{ij}(r_{(2)} h_{(1)}) d\tau_{ji}(r_{(3)} h_{(2)}))(1 \otimes h_{(3)})\\ &\quad = \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}))(1 \otimes h)\\ &\qquad - \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(h_{(1)}) \tau_{ji}(h_{(3)}) \pi^{j}_{i\Gamma_m}(A_{l_j}(r h_{(2)})))(1 \otimes h_{(4)})\\ &\qquad - \sum \iota^{i}_{ij\Gamma_m}(a \tau_{ij}(r_{(2)}) d\tau_{ji}(r_{(1)}))(1 \otimes h) = 0.\end{aligned}$$

The last identity comes from the fact that R is a right ideal.

Let $(\gamma_i)_{i \in I} \in \Gamma^1_{\mathrm{c}}(\mathcal{P})$. We have to prove that

$$(\pi^{i}_{j} \otimes id)_{\Gamma} \circ \mathrm{hor}_{l_i}(\gamma_i) = \phi_{ij\Gamma} \circ (\pi^{j}_{i} \otimes id)_{\Gamma} \circ \mathrm{hor}_{l_j}(\gamma_j).$$

γ_i has the general form

$$\gamma_i = \sum_k \chi_i(f^0_k) d\chi_i(f^1_k), \qquad f^0_k, f^1_k \in \mathcal{P}.$$

Using the gluing condition of (34) and (37) one verifies that γ_j has the form

$$\gamma_j = \sum_k \chi_j(f^0_k) d\chi_j(f^1_k) + \rho, \qquad \rho \in \chi_{j\Gamma}(\ker \chi_{i\Gamma}).$$

Now, one obtains from Lemma 5, (73) and (102)

$$
\begin{aligned}
&(\pi^i_j \otimes id)_\Gamma \circ \mathrm{hor}_{l_i}(\gamma_i)\\
&\quad = (\pi^i_j \otimes id)_\Gamma \circ \mathrm{hor}_{l_i}\left(\sum_k \chi_i(f^0_k)d\chi_i(f^1_k)\right)\\
&\quad = \sum_k (\pi^i_j \otimes id)_\Gamma(\chi_i(f^0_k)D_{l_i}(\chi_i(f^1_k)))\\
&\quad = \phi_{ij\Gamma} \circ (\pi^j_i \otimes id)_\Gamma \circ \left(\sum_k \chi_j(f^0_k)D_{l_j}(\chi_j(f^1_k))\right)\\
&\quad = \sum_k \phi_{ij\Gamma} \circ (\pi^j_i \otimes id)_\Gamma \circ \mathrm{hor}_{l_j}(\chi_j(f^0_k)d\chi_j(f^1_k) + \rho)\\
&\quad = \sum_k \phi_{ij\Gamma}(\pi^j_i \otimes id)_\Gamma \circ \mathrm{hor}_{l_j}(\gamma_j),
\end{aligned}
$$

and the assertion is proved. □

Proposition 15. *There exists a bijection between left and right connections.*

Proof. A left connection corresponds to a family of linear maps $(A_{l_i})_{i\in I}$ satisfying (70), (71) and (98). The linear maps $A_{r_i} := -A_{l_i} \circ S$ fulfill (70) and (99), thus the A_{r_i} define right connections on the trivilizations. One has to prove that the family $(A_{r_i})_{i\in I}$ satisfies (71). Using $\tau_{ij} \circ S = \tau_{ji}$ and $\sum d(\tau_{ij}(h_1)\tau_{ji}(h_2)) = 0$, one calculates

$$
\begin{aligned}
\pi^i_{j_{\Gamma_m}}(A_{r_i}(h)) &= -\pi^i_{j_{\Gamma_m}}(A_{l_i}(S(h)))\\
&= -\sum \tau_{ij}(S(h_{(3)}))\pi^j_{i_{\Gamma_m}}(A_{l_j}(S(h_{(2)})))\tau_{ji}(S(h_{(1)}))\\
&\quad -\sum \tau_{ij}(S(h_{(2)}))d\tau_{ji}(S(h_{(1)}))\\
&= -\sum \tau_{ji}(h_{(3)})\pi^j_{i_{\Gamma_m}}(A_{l_j}(S(h_{(2)})))\tau_{ij}(h_{(1)})\\
&\quad -\sum \tau_{ji}(h_{(2)})d\tau_{ij}(h_{(1)})\\
&= -\sum \tau_{ij}(h_{(1)})\pi^j_{i_{\Gamma_m}}(A_{l_j}(S(h_{(2)})))\tau_{ji}(h_{(3)}) + \sum \tau_{ij}(h_{(1)})d\tau_{ji}(h_{(2)})\\
&= \sum \tau_{ij}(h_{(1)})\pi^j_{i_{\Gamma_m}}(A_{r_j}(h_{(2)}))\tau_{ji}(h_{(3)}) + \sum \tau_{ij}(h_{(1)})d\tau_{ji}(h_{(2)}).
\end{aligned}
$$
□

Remark. A left (right) connection $\mathrm{hor}_{l,r}$ and the corresponding left (right) covariant derivatives $D_{l,r}$ are connected by $\mathrm{hor}_{l,r} \circ d = D_{l,r}|_{\mathcal{P}}$. Note that $\mathrm{hor}_{l,r}$ can be extended to the submodule

$$\Pi(\mathcal{P}) := \{\gamma \in \Gamma_{\mathrm{c}}(\mathcal{P}) | \chi_{i_{\Gamma_{\mathrm{c}}}}(\gamma) \in (\Gamma(B_i)\hat{\otimes}H) \oplus (\Gamma(B_i)\hat{\otimes}\Gamma^1(H))\}.$$

This means that the equation

$$\mathrm{hor}_{l,r} \circ d = D_{l,r}$$

is valid on hor $\Gamma_{\mathrm{c}}(\mathcal{P})$).

To discuss curvatures of covariant derivatives and connections, we introduce the notion of left (right) pre-connection forms.

Definition 8. A left (right) pre-connection form $\omega_{l,r}$ is a linear map $\omega_{l,r}: H \to \Gamma^1_{\mathrm{c}}(\mathcal{P})$ satisfying

$$\omega_{l,r}(1) = 0, \tag{108}$$

$$\Delta^{\Gamma}_{\mathcal{P}}(\omega_l(h)) = \sum \omega_l(h_{(2)}) \otimes S(h_{(1)})h_{(3)}, \tag{109}$$

$$\Delta^{\Gamma}_{\mathcal{P}}(\omega_r(h)) = \sum \omega_r(h_{(2)}) \otimes h_{(3)}S^{-1}(h_{(1)}), \tag{110}$$

$$(1 - \mathrm{hor_c}) \circ \chi_{i_{\Gamma_c}}(\omega_l(h)) = -\sum 1 \hat{\otimes} S(h_{(1)})dh_{(2)} \quad \forall i \in I, \tag{111}$$

$$(1 - \mathrm{hor_c}) \circ \chi_{i_{\Gamma_c}}(\omega_r(h)) = -\sum 1 \hat{\otimes} (dh_{(2)})S^{-1}(h_{(1)}) \quad \forall i \in I. \tag{112}$$

Proposition 16. *Left* (*right*) *covariant derivatives are in bijective correspondence to left* (*right*) *pre-connection forms*.

Proof. Let ω_l be a left pre-connection form ω_l determines a family of linear maps A_{l_i} by

$$A_{l_i}(h) := -(id \otimes \varepsilon) \circ \mathrm{hor_c} \circ \chi_{i_{\Gamma_c}}(\omega_l(h)). \tag{113}$$

Because of (108), the A_{l_i} fulfill (70).

Using

$$(1 - \mathrm{hor_c}) \circ \chi_{i_{\Gamma_c}}(\omega_l(h)) + \mathrm{hor_c} \circ \chi_{i_{\Gamma_c}}(\omega_l(h)) = \chi_{i_{\Gamma_c}}(\omega_l(h)),$$

(109) and (111), formula (75) and (113) one verifies easily

$$\chi_{i_{\Gamma_c}}(\omega_l(h)) = -\sum 1 \hat{\otimes} S(h_{(1)})dh_{(2)} - \sum A_{l_i}(h_{(2)}) \hat{\otimes} S(h_{(1)})h_{(3)}. \tag{114}$$

Since

$$(\pi^i_j \otimes id) \circ \chi_{i_{\Gamma_c}}(\omega_l(h)) = \phi_{ij\Gamma} \circ (\pi^j_i \otimes id) \circ \chi_{j_{\Gamma_c}}(\omega_l(h)),$$

an easy calculation (using (15) and the projection P_{inv} (85)) leads to (71).

We want to prove that the left-covariant derivative D_l determined by the A_{l_i} is

$$D_l(\gamma) = d\gamma + (-1)^n \sum \gamma_{(0)}\omega_l(\gamma_{(1)}), \quad \gamma \in \mathrm{hor}\,\Gamma^n_{\mathrm{c}}(\mathcal{P}). \tag{115}$$

It is sufficient to prove that for $\gamma \in \mathrm{hor}\,\Gamma^n_{\mathrm{c}}(\mathcal{P})$

$$\begin{aligned}\chi_{i_{\Gamma_c}}(d\gamma + (-1)^n \sum \gamma_{(0)}\omega_l(\gamma_{(1)})) &= d\gamma \hat{\otimes} h + (-1)^{n+1}\gamma A_{l_i}(h_{(1)}) \hat{\otimes} h_{(2)} \\ &= \chi_{i_{\Gamma_c}} \circ D_l(\gamma).\end{aligned}$$

$\chi_{i_{\Gamma_c}}(\gamma)$ has the general form

$$\chi_{i_{\Gamma_c}}(\gamma) = \sum_k \gamma^k_i \hat{\otimes} h^k_i, \quad \gamma^k_i \in \Gamma^n(B_i), \quad h^k_i \in H.$$

Using (114) one obtains

$$\begin{aligned}
&\chi_{i_{\Gamma_c}}\left(d\gamma + (-1)^n \sum \gamma_{(0)}\omega_l(\gamma_{(1)})\right)\\
&= \left(\sum_k d\gamma_i^k \hat{\otimes} h_i^k + (-1)^n \sum_k \gamma_i^k \hat{\otimes} dh_i^k + (-1)^n \sum_k \sum (\gamma_i^k \otimes h^k_{i(1)})\chi_{i_{\Gamma_c}}(\omega_l(h^k_{i(2)})\right)\\
&= \sum_k d\gamma_i^k \hat{\otimes} h_i^k + (-1)^n \sum_k \gamma_i^k \hat{\otimes} dh_i^k - (-1)^n \sum_k \sum (\gamma_i^k \otimes h^k_{i(1)})(1\hat{\otimes} S(h^k_{i(2)})dh^k_{i(3)})\\
&\quad -(-1)^n \sum_k \sum (\gamma_i^k \otimes h^k_{i(1)})(A_{l_i}(h^k_{i(3)})\hat{\otimes} S(h^k_{i(2)})dh^k_{i(4)})\\
&= d\gamma\hat{\otimes} h + (-1)^{n+1}\gamma A_{l_i}(h_{(1)})\hat{\otimes} h_{(2)}.
\end{aligned}$$

Note that the following identity is satisfied:

$$\begin{aligned}
D_{l_i}(\gamma\hat{\otimes} h) &= d\gamma\hat{\otimes} h + (-1)^{n+1}\sum \gamma A_{l_i}(h_{(1)})\hat{\otimes} h_{(2)}\\
&= d(\gamma\hat{\otimes} h) + (-1)^n \sum (\gamma\hat{\otimes} h_{(1)})\chi_{i_{\Gamma_c}}(\omega_l(h_{(2)})).
\end{aligned} \tag{116}$$

Assume now that there is given a left-covariant derivative D_l. In terms of the corresponding linear maps A_{l_i}, one obtains a family of left pre-connection forms $\omega_{l_i} : H \to (\Gamma(B_i)\hat{\otimes}\Gamma(H))^1$ by

$$\omega_{l_i}(h) = -\sum 1\hat{\otimes} S(h_{(1)})dh_{(2)} - \sum A_{l_i}(h_{(2)})\hat{\otimes} S(h_{(1)})h_{(3)}.$$

Using (71), one obtains

$$(\pi^i_j \otimes id)_\Gamma(\omega_{l_i}(h)) = \phi_{ij_\Gamma} \circ (\pi^j_i \otimes id)_\Gamma(\omega_{l_j}(h),$$

thus

$$\omega_l(h) := (\omega_{l_i}(h))_{i\in I}$$

defines a left pre-connection form $\omega_l : H \to \Gamma_c^1(\mathcal{P})$.

One easily verifies for $\gamma\hat{\otimes} h \in \Gamma^n(B_i)\hat{\otimes} H$

$$\begin{aligned}
D_{l_i}(\gamma\hat{\otimes} h) &= d\gamma\hat{\otimes} h + (-1)^{n+1}\sum \gamma A_{l_i}(h_{(1)})\hat{\otimes} h_{(2)}\\
&= d(\gamma\hat{\otimes} h) + (-1)^n \sum (\gamma\hat{\otimes} h_{(1)})\omega_{l_i}(h_{(2)}).
\end{aligned} \tag{117}$$

Using this formula it follows that

$$D_l(\gamma) = d\gamma + (-1)^n \sum \gamma_{(0)}\omega_l(\gamma_{(1)}) \tag{118}$$

for $\gamma \in \mathrm{hor}\,\Gamma_c^n(\mathcal{P})$. It is immediate from the formulas (116) and (117) and Proposition 12 that the correspondence is bijective.

For right-covariant derivatives the proof is analogous. □

Remark. Note that the foregoing proof also shows the bijective correspondence between left (right) pre-connection forms and families of linear maps $A_{l,r_i} : H \to \Gamma(B_i)$ fulfilling (70) and (71).

Definition 9. A left (right) pre-connection form $\omega_{l,r}$ is called left (right) connection form, if

$$R \subset \ker((id \otimes \varepsilon) \circ \mathrm{hor}_{\mathrm{c}} \circ \chi_{i_{\Gamma_{\mathrm{c}}}} \circ \omega_l) \quad \forall i \in I, \tag{119}$$

$$S^{-1}(R) \subset \ker((id \otimes \varepsilon) \circ \mathrm{hor}_{\mathrm{c}} \circ \chi_{i_{\Gamma_{\mathrm{c}}}} \circ \omega_r) \quad \forall i \in I \tag{120}$$

is satisfied.

Proposition 17. *Left (right) connections are in bijective correspondence to left (right) connection forms.*

Proof. The claim follows immediately from Propositions 14 and 16 and (113). □

Remark. Note that classical connection forms are related to the connection forms considered above as follows. Let a classical principal bundle with total space Q and structure group G be given. A classical connection form is a Lie algebra valued 1-form $\tilde{\omega}$ of type Ad on Q. Let X be a vector field on Q, and let $h \in C^\infty(G)$. Then, the formula

$$\omega_l(h)(X) = -\tilde{\omega}(X)(h)$$

defines a left connection form ω_l in the above sense. Condition (119) with $R = (\ker \varepsilon)^2$ means that $\tilde{\omega}$ can be interpreted as a Lie algebra valued form. In this case (109) and (111) replace the usual conditions (type Ad, condition for fundamental vectors) for connection forms.

Definition 10. The left (right) curvature of a given left (right) covariant derivative is the linear map $D^2_{l,r} : \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P}) \to \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P})$.

Definition 11. Let $\omega_{l,r}$ be a left (right) pre-connection form of a left (right) covariant derivative $D_{l,r}$. The linear maps $\Omega_{l,r} : H \to \Gamma^2_{\mathrm{c}}(\mathcal{P})$ defined by

$$\Omega_l(h) := d\omega_l(h) - \sum \omega_l(h_{(1)})\omega_l(h_{(2)}), \tag{121}$$

$$\Omega_r(h) := d\omega_r(h) + \sum \omega_r(h_{(2)})\omega_r(h_{(1)}) \tag{122}$$

are called the left (right) curvature form of a given left (right) covariant derivative.

Remark. In other words, we take an analogue of the structure equation as definition of the curvature form.

Proposition 18. *The left (right) curvature of a given left (right) covariant derivative is related to the left (right) curvature form by the identity*

$$D_l^2(\gamma) = \sum \gamma_{(0)} \Omega_l(\gamma_{(1)}), \quad \gamma \in \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P}), \tag{123}$$

$$D_r^2(\gamma) = \sum \Omega_r(\gamma_{(1)})\gamma_{(0)}, \quad \gamma \in \mathrm{hor}\, \Gamma_{\mathrm{c}}(\mathcal{P}). \tag{124}$$

Proof. Because of the one-to-one correspondence between covariant derivatives on $\mathcal{P}$ and certain families of covariant derivatives on the trivilizations $B_i \otimes H$, it is sufficient to prove this assertion on a trivial bundle $B \otimes H$. In this case, the linear map ω_l belonging to a left-covariant derivative has the form

$$\omega_l(h) = -\left(1 \otimes \sum S(h_{(1)})dh_{(2)}\right) - \sum (A_l(h_{(2)}) \hat{\otimes} S(h_{(1)})h_{(3)}).$$

Therefore, one obtains for Ω_l

$$\begin{aligned}
& d\omega_l(h) - \sum \omega_l(h_{(1)})\omega_l(h_{(2)}) \\
& = -1\hat{\otimes} \sum dS(h_{(1)})dh_{(2)} - \sum dA_l(h_{(2)}) \hat{\otimes} S(h_{(1)})h_{(3)} + \sum A_l(h_{(2)}) \hat{\otimes} (dS(h_{(1)}))h_{(3)} \\
& \quad + \sum A_l(h_{(2)}) \hat{\otimes} S(h_{(1)})dh_{(3)} - 1\hat{\otimes} \sum S(h_{(1)})(dh_{(2)})S(h_{(3)})dh_{(4)} \\
& \quad - \sum A_l(h_{(2)}) \hat{\otimes} S(h_{(1)})h_{(3)}S(h_{(4)})dh_{(5)} + \sum A_l(h_{(4)}) \hat{\otimes} S(h_{(1)})(dh_{(2)})S(h_{(3)})h_{(5)} \\
& \quad - \sum A_l(h_{(2)})A_l(h_{(5)}) \hat{\otimes} S(h_{(1)})h_{(3)}S(h_{(4)})h_{(6)} \\
& = -\sum dA_l(h_{(2)}) \hat{\otimes} S(h_{(1)})h_{(3)} - \sum A_l(h_{(2)})A_l(h_{(3)}) \hat{\otimes} S(h_{(1)})h_{(4)},
\end{aligned}$$

which leads for $\gamma \in \Gamma^n(B)$ to

$$\sum (\gamma \hat{\otimes} h_{(1)})\Omega_l(h_{(2)}) = -\sum \gamma dA_l(h_{(1)}) \hat{\otimes} h_{(2)} - \sum \gamma A_l(h_{(1)})A_l(h_{(2)}) \hat{\otimes} h_{(3)}.$$

On the other hand, the left-hand side of (123) is

$$\begin{aligned}
(D_l)^2(\gamma \otimes h) &= D_l\left(d\gamma \hat{\otimes} h - \sum (-1)^n \gamma A_l(h_{(1)}) \hat{\otimes} h_{(2)}\right) \\
&= -(-1)^{n+1} \sum (d\gamma)A_l(h_{(1)}) \hat{\otimes} h_{(2)} - (-1)^n \sum (d\gamma)A_l(h_{(1)}) \hat{\otimes} h_{(2)} \\
&\quad - \sum \gamma dA_l(h_{(1)}) \hat{\otimes} h_{(2)} - \sum \gamma A_l(h_{(1)})A_l(h_{(2)}) \hat{\otimes} h_{(3)} \\
&= -\left(\sum \gamma dA_l(h_{(1)}) \hat{\otimes} h_{(2)} + \sum \gamma A_l(h_{(1)})A_l(h_{(2)} \hat{\otimes} h_{(3)}\right) \\
&= \sum (\gamma \hat{\otimes} h_{(1)})\Omega_l(h_{(2)}).
\end{aligned}$$

For right-covariant derivatives the proof is analogous. □

Remark. The proof shows that there is a linear map $F_{l,r} : H \to \Gamma^2(B)$ defined by

$$F_l(h) := dA_l(h) + \sum A_l(h_{(1)})A_l(h_{(2)}), \tag{125}$$

$$F_r(h) := dA_r(h) - \sum A_r(h_{(2)})A_r(h_{(1)}) \tag{126}$$

which is related to the left (right) curvature form of a given left (right) covariant derivative on a trivial QPFB by

$$\Omega_l(h) = -\sum F_l(h_{(2)})\hat{\otimes}S(h_{(1)})h_{(3)}, \tag{127}$$

$$\Omega_r(h) = -\sum F_r(h_{(2)})\hat{\otimes}h_{(3)}S^{-1}(h_{(1)}). \tag{128}$$

In the case of a general locally trivial bundle one arrives at a family $F_{l,r_i} : H \to \Gamma^2(B_i)$ of linear maps, called local curvature forms, which are related to the local connection forms A_{l,r_i} by (125) and (126). Using formula (71) and the Leibniz rule (taking into account $\sum \tau_{ij}(h_{(1)})\tau_{ij}(h_{(2)}) = \varepsilon(h)1$) it is easy to verify that the local curvature forms satisfy

$$\pi^i_{j_{\Gamma_m}}(F_{l,r_i}(h)) = \sum \tau_{ij}(h_{(1)})\pi^j_{i_{\Gamma_m}}(F_{l,r_j}(h_{(2)}))\tau_{ji}(h_{(3)}). \tag{129}$$

An analogue of the Bianchi identity does in general not exist.

Now, we make some remarks about the general form of the linear maps $A_{l_i} : H \to \Gamma^1(B_i)$ corresponding to connections on a locally trivial QPFB. For this we use the functionals $\mathcal{X}_i$ corresponding to the right ideal R, which determines the right-covariant differential calculus $\Gamma(H)$ (see [26,27] and Appendix A). Let the $h^i + R \in \ker\varepsilon/R$ be a linear basis in $\ker\varepsilon/R$. Then every element $h - \varepsilon(h)1 + R \in \ker\varepsilon/R$ has the form $\mathcal{X}_i(h)h^i + R$. Since $1 \in \ker A_{l_i}$ and $R \subset \ker A_{l_i}$ it follows that A_{l_i} is determined by its values on the h^k,

$$A_{l_i}(h) = \sum_k \mathcal{X}_k(h)A_{l_i}(h^k).$$

In other words, to get a connection on the trivial pieces $B_i \otimes H$, one chooses $A^k_i \in \Gamma^1(B_i)$ and defines the linear map A_{l_i} by

$$A_{l_i}(h) = \sum_k \mathcal{X}_k(h)A^k_i.$$

The connections so defined on the trivial pieces $B_i \otimes H$ do in general not give a connection on the locally trivial QPFB $\mathcal{P}$, because they do in general not fulfill the condition (71). If the right ideal R fulfills (49), one can rewrite the condition (71) as a condition for the one-forms $A^k_i \in \Gamma^1(B_i)$. Recall that in this case $\sum \tau_{ij}(r_{(1)})d\tau_{ji}(r_{(2)}) = 0 \quad \forall r \in R$ (cf. (51)), thus

$$\sum \tau_{ij}(h_{(1)})d\tau_{ij}(h_{(2)}) = \sum\sum_k \mathcal{X}_k(h)\tau_{ij}(h^k_{(1)})d\tau_{ji}(h^k_{(2)}).$$

Furthermore, the condition (49) leads to the identity

$$\begin{aligned}\sum \tau_{ij}(h_{(1)})\mathcal{X}_l(h_{(2)})\tau_{ji}(h_{(3)}) &= \sum \tau_{ji}(S(h_{(1)})h_{(3)})\mathcal{X}_l(h_{(2)})\\ &= \sum_k \mathcal{X}_k(h)\sum \tau_{ji}(S(h^k_{(1)})h^k_{(3)})\mathcal{X}_l(h^k_{(2)}).\end{aligned}$$

Putting now $A_{l_i}(h) = \sum_k \mathcal{X}_k(h) A_i^k$ in (71) leads to the following condition for the forms A_i^k:

$$\pi^i_{j_{\Gamma_m}}(A_i^k) = \sum_l \tau_{ji}(S(h^k_{(1)})h^k_{(3)})\mathcal{X}_l(h^k_{(2)})\pi^j_{i_{\Gamma_m}}(A_i^l) + \tau_{ij}(h^k_{(1)})d\tau_{ji}(h^k_{(2)}).$$

Note that, in the case $I = \{1, 2\}$, it follows from the last formula that there exist connections. One can choose, e.g. one-forms A_2^l on the right, and solve the remaining equation for A_1^k due to the surjectivity of $\pi^1_{2_{\Gamma_m}}$.

One can regard the set $\mathcal{D}_{l,r}$ of all left (right) covariant derivatives as a set with affine structure, where the corresponding vector space is characterized by the following proposition.

Proposition 19. *A linear map* $C_{l,r} : \mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P}) \to \mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P})$ *is a difference of two left* (*right*) *covariant derivatives if and only if*

$$C_{l,r}(1) = 0, \tag{130}$$

$$C_{l,r}(\mathrm{hor}\,\Gamma^n_{\mathrm{c}}(\mathcal{P})) \subset \mathrm{hor}\,\Gamma^{n+1}_{\mathrm{c}}(\mathcal{P}), \tag{131}$$

$$C_l(\gamma\alpha) = (-1)^n \gamma C_l(\alpha), \quad \gamma \in \Gamma^n_{\mathrm{c}}(B), \ \alpha \in \mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P}), \tag{132}$$

$$C_r(\alpha\gamma) = (-1)^n C_r(\alpha)\gamma, \quad \gamma \in \Gamma_{\mathrm{c}}(B), \ \alpha \in \mathrm{hor}\,\Gamma^n_{\mathrm{c}}(\mathcal{P}), \tag{133}$$

$$(C_{l,r} \otimes id) \circ \Delta_{\mathcal{P}_{\Gamma_{\mathrm{c}}}} = \Delta_{\mathcal{P}_{\Gamma_{\mathrm{c}}}} \circ C_{l,r}, \tag{134}$$

$$C_{l,r}(\ker \chi_{i_{\Gamma_{\mathrm{c}}}}|_{\mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P})}) \subset \ker \chi_{i_{\Gamma_{\mathrm{c}}}}|_{\mathrm{hor}\,\Gamma_{\mathrm{c}}(\mathcal{P})} \quad \forall i \in I. \tag{135}$$

This is immediate from Definition 6.

Because of (135) such a map $C_{l,r}$ defines a family of local maps C_{l,r_i} by

$$C_{l,r_i} \circ \chi_{i_{\Gamma_{\mathrm{c}}}} = \chi_{i_{\Gamma_{\mathrm{c}}}} \circ C_{l,r}.$$

It is immediate that the set of left (right) connections is an affine subspace of $\mathcal{D}_{l,r}$. The elements of the corresponding vector space have the following additional property:

$$(id \otimes \varepsilon) \circ C_{l_i}(1 \otimes r) = 0, \quad \forall i \in I, \quad \forall r \in R,$$

$$(id \otimes \varepsilon) \circ C_{r_i}(1 \otimes r) = 0, \quad \forall i \in I, \quad \forall r \in S^{-1}(R).$$

5. Example

Here we present an example of a $U(1)$-bundle over the quantum space $S^2_{pq\phi}$. The quantum space $S^2_{pq\phi}$ is treated in detail in [6] and we restrict ourselves here to a brief summary.

The algebra $P(S^2_{pq\phi})$ of all polynomials over the quantum space $S^2_{pq\phi}$ is constructed by gluing together two copies of a quantum disc along its classical subspace.

Definition 12 (cf. [16]). The algebra $P(D_p)$ of all polynomials over the quantum disc D_p is defined as the algebra generated by the elements x and x^* fulfilling the relation

$$x^*x - pxx^* = (1-p)1, \tag{136}$$

where $0 < p < 1$.

$P(D_p)$ is a *-algebra in a natural way. Let $P(S^1)$ be the algebra generated by the elements u, u^* fulfilling the relation

$$uu^* = u^*u = 1.$$

$P(S^1)$ can be considered as the algebra of all trigonometrical polynomials over the circle S^1. There exists a surjective *-homomorphism $\phi_p : P(D_p) \to P(S^1)$ defined by

$$\phi_p(x) = u, \tag{137}$$

which can be considered as the "pull back" of the embedding of the circle into the quantum disc. The algebra $P(S^2_{pq\phi})$ of all polynomials over the quantum space $S^2_{pq\phi}$ is defined as

$$P(S^2_{pq\phi}) := \{(f,g) \in P(D_p) \oplus (D_q) | \phi_p(f) = \phi_q(g)\} \tag{138}$$

with $0 < p, q < 1$. It was shown in [6] that this algebra can be regarded as the algebra generated by the elements f_1, f_{-1} and f_0 fulfilling the relations

$$f_{-1}f_1 - qf_1f_{-1} = (p-q)f_0 + (1-p)1, \tag{139}$$

$$f_0f_1 - pf_1f_0 = (1-p)f_1, \tag{140}$$

$$f_{-1}f_0 - pf_0f_{-1} = (1-p)f_{-1}, \tag{141}$$

$$(1-f_0)(f_1f_{-1} - f_0) = 0, \tag{142}$$

where the isomorphism is given by $f_1 \mapsto (x, y)$, $f_{-1} \mapsto (x^*, y^*)$ and $f_0 \mapsto (xx^*, 1)$. (Here, the generators of $P(D_q)$ are denoted by y and y^*.) As was proved in [6], the C^*-closure $C(S^2_{pq\phi})$ of $P(S^2_{pq\phi})$ (formed using representations in bounded operators) is isomorphic to the C^*-algebra $C(S^2_{\mu c})$ of the Podles sphere $S^2_{\mu c}$ for $c > 0$.

Now, let us construct a class of QPFBs with structure group $U(1)$ and base space $S^2_{pq\phi}$. The algebra of polynomials $P(U(1))$ over $U(1)$ by definition coincides with the algebra $P(S^1)$. With $\Delta(u) = u \otimes u$, $\varepsilon(u) = 1$ and $S(u) = u^*$, $P(U(1))$ is a Hopf algebra. According to Proposition 4, we need just one transition function $\tau_{12} : P(U(1)) \to P(S^1)$ to obtain a locally trivial QPFB. We define transition functions $\tau_{12}^{(n)}$, $n = 0, 1, \ldots$ as follows:

$$\tau_{12}^{(n)}(u) := u^n, \qquad \tau_{12}^{(n)}(u^*) := u^{*n}.$$

It follows that

$$\tau_{21}^{(n)}(u) = u^{*n}, \qquad \tau_{21}^{(n)}(u^*) = u^n.$$

We obtain locally trivial QPFBs $(\mathcal{P}^{(n)}, \Delta_{\mathcal{P}^{(n)}}, P(U(1)), P(S^2_{pq\phi}), \iota, ((\chi_p, \ker\pi_p), (\chi_q, \ker\pi_q)))$ corresponding to these transition functions (see formulas (15) and (16)), where ι is the canonical embedding $P(S^2_{pq\phi}) \subset \mathcal{P}^{(n)}$ and $\pi_{p,q} : P(S^2_{pq\phi}) \to P(D_{p,q})$ and $\chi_{p,q} : \mathcal{P}^{(n)} \to P(D_{p,q}) \otimes P(U(1))$ are the restrictions of the canonical projections to $P(S^2_{pq\phi})$ and $\mathcal{P}^{(n)}$, respectively.

Proposition 20. *The locally trivial QPFBs $\mathcal{P}^{(n)}$ and $\mathcal{P}^{(m)}$ are nonisomorphic for $n \neq m$.*

Proof. Assume $\mathcal{P}^{(n)} \simeq \mathcal{P}^{(m)}$. According to Proposition 5, it follows that there exist homomorphisms $\sigma_1 : P(U(1)) \to P(D_p)$, $\sigma_2 : P(U(1)) \to P(D_q)$ with values in the centres of $P(D_{p,q})$ such that Eq. (21) is true, in particular, for the generator u. □

Lemma 6. *The centre of $P(D_p)$ is trivial, $Z(P(D_p)) = \mathbb{C}1$.*

Proof of the lemma. We make use of the fact that $P(D_p)$ has a vector space basis $(x^k x^{*l})_{k\geq 0, l>0}$ (see [6, Lemma 2]). Thus, any $f \in P(D_p)$ can be written as $f = f_- + \sum_{k=0}^{n} f_k x^k x^{*n-k}$, where f_- is a linear combination of elements $x^k x^{*l}$ with $k+l < n$. Now, the assumption $xf = fx$, together with the relation (136), immediately leads to $f_k = p^{n-k} f_k$, $k = 0, \ldots, n$, i.e. $f_k = 0$ for $k = 0, \ldots, n-1$. Analogously, the assumption $x^* f = fx^*$ yields $f_k = p^k f_k$, $k = 0, \ldots, n$, i.e. $f_k = 0$ for $k = 1, \ldots, n$. Thus, if f is in the centre, $f_k = 0$ for $k = 0, \ldots, n$. Repeating the argument, the degree of f is reduced to 0, which proves the lemma. □

Proof of the proposition (continued). Due to the lemma, the homomorphisms σ_1 and σ_2 can be considered as characters of $P(U(1))$. Writing now (21) for $h = u$, $\tau_{12}^{(n)}$ on the left and $\tau_{12}^{(m)}$ on the right leads to $u^m = \lambda u^n$, $\lambda \in \mathbb{C}$, which is possible only for $m = n$, since the powers of u form a basis of $P(S^1)$. The proposition is proved. □

We remind the reader that we assumed $0 < p, q < 1$, so that p, q cannot be roots of unity, which makes the arguments in the above proofs meaningful.

Note that $\mathcal{P}^{(0)}$ is the trivial bundle $P(S^2_{pq\phi}) \otimes P(U(1))$. Due to the foregoing proposition, all the other $\mathcal{P}^{(n)}$ are nontrivial.

In the following, we restrict ourselves to the case $n = 1$.

Proposition 21. *Let $\bar{J} \subset P(D_p) \otimes P(D_q)$ be the ideal generated by the element*

$$(xx^* - 1) \otimes (yy^* - 1).$$

Moreover, let J be the ideal in the free algebra $\mathbb{C}\langle a, a^*, b, b^*\rangle$ generated by the relations

$$a^*a - qaa^* = (1-q)1, \tag{143}$$

$$b^*b - pbb^* = (1-p)1, \tag{144}$$

$$ba = ab, \qquad ba^* = a^*b, \qquad b^*a = ab^*, \qquad b^*a^* = a^*b^*, \tag{145}$$

$$(1 - aa^*)(1 - bb^*) = 0. \tag{146}$$

Then, we have the isomorphisms of **-algebras* $\mathcal{P}^{(1)} \simeq (P(D_p) \otimes P(D_q))/\bar{J} \simeq \mathbb{C}\langle a, a^*, b, b^*\rangle/J$.

Proof. $(P(D_p) \otimes P(D_q))/J$ is generated by

$$\bar{a} = 1 \otimes y + J, \qquad \bar{a}^* = 1 \otimes y^* + J, \qquad \bar{b} = x \otimes 1 + J, \qquad \bar{b}^* = x^* \otimes 1 + J.$$

It is easy to see that $\bar{a} \mapsto a$, $\bar{b} \mapsto b$ defines an isomorphism $(P(D_p) \otimes P(D_q))/\bar{J} \to \mathbb{C}\langle a, a^*, b, b^*\rangle/J$. It remains to prove a second isomorphy, e.g. $\mathcal{P}^{(1)} \simeq \mathbb{C}\langle a, a^*, b, b^*\rangle/J$. Consider the following elements in $\mathcal{P}^{(1)}$:

$$\tilde{a} = (1 \otimes u, y \otimes u), \qquad \tilde{a}^* = (1 \otimes u^*, y^* \otimes u^*),$$
$$\tilde{b} = (x \otimes u^*, 1 \otimes u^*), \qquad \tilde{b}^* = (x^* \otimes u, 1 \otimes u).$$

A short calculation shows that these elements fulfill the same relations (145) as the a, a^*, b and b^*. Thus, there exists a homomorphism $F : \mathbb{C}\langle a, a^*, b, b^*\rangle/J \to \mathcal{P}^{(1)}$ defined by

$$F(a) := \tilde{a}, \qquad F(b) := \tilde{b}, \qquad F(a^*) := \tilde{a}^*, \qquad F(b^*) := \tilde{b}^*.$$

We will show that F is an isomorphism. For surjectivity it is sufficient to show that the elements $\tilde{a}, \tilde{a}^*, \tilde{b}$ and $\tilde{b}^*$ generate the algebra $\mathcal{P}^{(1)}$. Again we use that the elements $x^k x^{*l}$, $k \geq 0$, $l > 0$ form a vector space basis of $P(D_p)$ ([6, Lemma 2]) and that the elements u^i, $i \in \mathbb{Z}$ ($u^{-1} = u^*$), form a vector space basis in $P(U(1))$. Thus a general element $f \in P(D_p) \otimes P(U(1)) \oplus P(D_q) \otimes P(U(1))$ has the form

$$f = \left(\sum_{k,l \geq 0, i \in \mathbb{Z}} c^p_{k,l,i} x^k x^{*l} \otimes u^i, \sum_{m,n \geq 0, j \in \mathbb{Z}} c^q_{m,n,j} y^m y^{*n} \otimes u^j \right).$$

$f \in \mathcal{P}^{(1)}$ means that there is the restriction

$$\sum_{k,l \geq 0, i \in \mathbb{Z}} c^p_{k,l,i} u^{k-l} \otimes u^i = \sum_{m,n \geq 0, j \in \mathbb{Z}} c^q_{m,n,j} u^{m-n-j} \otimes u^j,$$

which leads to the following condition for the coefficients $c^p_{k,l,i}$ and $c^q_{m,n,j}$:

$$\sum_{l \geq 0, s+l \geq 0} c^p_{s+l,l,t} = \sum_{n \geq 0, n+s+t \geq 0} c^q_{s+t+n,n,t} \quad \forall s, t \in \mathbb{Z}. \tag{147}$$

$f \in \mathcal{P}^{(1)}$ has the form $f = \sum_{s,t} f_{s,t}$, where

$$f_{s,t} = \left(\sum_{l \geq 0, l+s \geq 0} c^p_{s+l,l,t} x^{l+s} x^{*l} \otimes u^t, \sum_{n \geq 0, n+s+t \geq 0} c^q_{s+t+n,n,t} y^{n+s+t} y^{*n} \otimes u^t \right) \in \mathcal{P}^{(1)}$$

due to (147). Because of (147) one can write $f_{s,t}$ as

$$f_{s,t} = \sum_{l\geq 0, l+s\geq 0} c^p_{s+l,l,t}(x^{l+s}x^{*l}\otimes u^t, y^{m+s+t}y^{*m}\otimes u^t) + \sum_{n\geq 0, n+s+t\geq 0} c^q_{s+t+n,n,t}(x^{k+s}x^{*k}\otimes u^t, y^{n+l+t}y^{*n}\otimes u^t) - \sum_{l\geq 0, l+s\geq 0} c^p_{s+l,l,t}(x^{k+s}x^{*k}\otimes u^t, y^{m+s+t}y^{*m}\otimes u^t).$$

The identity

$$(x^s x^l x^{*l}\otimes u^t, y^{s+t}y^n y^{*n}\otimes u^t) = \tilde{a}^{s+t+n}\tilde{a}^{*n}\tilde{b}^{s+l}\tilde{b}^{*l},$$

which is a direct consequence of the definition of $\tilde{a}, \tilde{a}^*, \tilde{b}$ and $\tilde{b}^*$, shows that F is surjective.

To show the injectivity of F, we define homomorphisms $F_{p,q} : \mathbb{C}\langle a, a^*, b, b^*\rangle/J \to P(D_{p,q})\otimes P(U(1))$ by $F_{p,q} := \chi_{p,q}\circ F$. Because of $\ker\chi_p \cap \ker\chi_q = \{0\}$, $\ker F = \{0\}$ if and only if $\ker F_p \cap \ker F_q = \{0\}$. First let us describe the ideals $\ker F_{p,q}$. Let I_p and I_q be the ideals generated by $1 - aa^*$ and $1 - bb^*$, respectively. From (145) it is immediate that the algebras $(\mathbb{C}\langle a, a^*, b, b^*\rangle/J)/I_{p,q}$ are isomorphic to $P(D_{p,q})\otimes P(U(1))$, where the isomorphism $(\mathbb{C}\langle a, a^*, b, b^*\rangle/J)/I_p \to P(D_p)\otimes P(U(1))$ is defined by $a\mapsto 1\otimes u$, $b\mapsto x\otimes 1$, and the isomorphism $(\mathbb{C}\langle a, a^*, b, b^*\rangle)/J)/I_q \to P(D_q)\otimes P(U(1))$ is defined by $a\mapsto y\otimes 1$, $b\mapsto 1\otimes u$.

Moreover, there are automorphisms $\tilde{F}_{p,q} : P(D_{p,q})\otimes P(U(1)) \to P(D_{p,q})\otimes P(U(1))$ defined by

$$\tilde{F}_p(1\otimes u) := 1\otimes u, \quad \tilde{F}_q(1\otimes u) := 1\otimes u, \quad \tilde{F}_p(1\otimes u^*) := 1\otimes u^*,$$
$$\tilde{F}_q(1\otimes u^*) := 1\otimes u^*, \quad \tilde{F}_p(x\otimes 1) := x\otimes u^*, \quad \tilde{F}_q(y\otimes 1) := y\otimes u,$$
$$\tilde{F}_p(x^*\otimes 1) := x^*\otimes u, \quad \tilde{F}_q(y^*\otimes 1) := y^*\otimes u^*.$$

Let $\eta_{p,q}$ be the quotient maps with respect to the ideals $I_{p,q}$. A short calculation shows that

$$F_{p,q} = \tilde{F}_{p,q}\circ\eta_{p,q},$$

which means $\ker F_{p,q} = I_{p,q}$. It remains to show $I_p \cap I_q = \{0\}$. There are the following identities in $\mathbb{C}\langle a, a^*, b, b^*\rangle/J$:

$$(1-aa^*)a = qa(1-aa^*), \quad (1-aa^*)a^* = q^{-1}a^*(1-aa^*),$$
$$(1-bb^*)b = pb(1-bb^*), \quad (1-bb^*)b^* = p^{-1}b^*(1-bb^*).$$

From these relations and the definition of $I_p = \ker F_p$ it follows that for $f \in \ker F_p$ there exists an element $\tilde{f}$ such that $f = (1-aa^*)\tilde{f}$. $\ker F_q$ has an analogous property with $1-bb^*$ instead of $1-aa^*$. Using that $1-xx^*$ is not a zero divisor in $P(D_p)$ (see [6, Lemma 3]), it is now easy to see that $f \in \ker F_p \cap \ker F_q$ is of the form $f = (1-aa^*)(1-bb^*)\tilde{f}$. Thus $f = 0$, i.e. $\ker F_p \cap \ker F_q = 0$. □

Using the identification $\mathcal{P}^{(1)} \simeq \mathbb{C}\langle a, a^*, b, b^*\rangle/J$, the mappings belonging to the bundle can be given explicitly

$$\begin{aligned}
&\Delta_{\mathcal{P}^{(1)}}(a) = a \otimes u, \qquad \Delta_{\mathcal{P}^{(1)}}(a^*) = a^* \otimes u^*, \qquad \Delta_{\mathcal{P}^{(1)}}(b) = b \otimes u^*,\\
&\Delta_{\mathcal{P}^{(1)}}(b^*) = b^* \otimes u, \qquad \chi_p(a) = 1 \otimes u, \qquad \chi_q(a) = y \otimes u,\\
&\chi_p(a^*) = 1 \otimes u^*, \qquad \chi_q(a^*) = y^* \otimes u^*, \qquad \chi_p(b) = x \otimes u^*,\\
&\chi_q(b) = 1 \otimes u^*, \qquad \chi_p(b^*) = x^* \otimes u, \qquad \chi_q(b^*) = 1 \otimes u,\\
&\iota(f_1) = ba, \qquad \iota(f_{-1}) = a^*b^*, \qquad \iota(f_0) = bb^*.
\end{aligned}$$

One easily finds the classical points, i.e. the characters, of the *-algebra $\mathbb{C}\langle a, a^*, b, b^*\rangle/J$. They are given by

$$\rho_{\theta_1\theta_2}(a) = \mathrm{e}^{\mathrm{i}\theta_1}, \tag{148}$$

$$\rho_{\theta_1\theta_2}(b) = \mathrm{e}^{\mathrm{i}\theta_2} \tag{149}$$

with $\theta_1, \theta_2 \in [0, 2\pi)$. Thus, the space of classical points can be identified with the two-torus T^2. As a consequence, the total space algebra of our $U(1)$-bundle is nonisomorphic to $P(SU_q(2))$, whose space of classical points is S^1. This nonisomorphy also extends to a possible C^*-closure. Therefore, our bundle is nonisomorphic to the quantum principal $U(1)$-bundle used by Brzeziński and Majid ([2,3]) whose total space algebra is $P(SU_q(2))$. This will remain true also if one goes to a C^*-completion, where the basis algebras would coincide. On the other hand, the bundle considered here is a natural analogue of the classical $U(1)$-Hopf bundle from a topological point of view. It is by definition a gluing of two quantum solid tori $S^1 \times D_{p,q}$ along their "boundary" T^2 of classical points, and the gluing on this boundary is exactly the same as in the classical case, formulated in terms of the pull-backs of the classical gluing maps.

We also note that the total space algebra $\mathcal{P}^{(1)}$ of our bundle seems not to have a Hopf algebra structure in an obvious way, which is also in contrast with the example of [2], where the total space algebra is $P(SU_q(2))$.

In the case $p = q = 1$, the algebra becomes commutative and only the relation $(1 - aa^*)(1 - bb^*) = 0$ remains. It is easy to see that this relation, together with the natural requirement $|a| \leq 1$, $|b| \leq 1$, describes a subspace of $\mathbb{R}^4$ homeomorphic to S^3. The right $U(1)$-action is a simultaneous rotation in a and b, and the orbit through $b = 0$ is the fibre over the top $(0, 0, 0)$ of the base space (see the discussion in [6]).

To build a connection on the locally trivial QPFB $\mathcal{P}^{(1)}$, first we have to construct an adapted covariant differential structure. By Definition 4, the adapted covariant differential structure is determined by differential calculi $\Gamma(P(D_p))$ and $\Gamma(P(D_q))$ and a right-covariant differential calculus $\Gamma(P(U(1))$ on the Hopf algebra $P(U(1))$.

As the differential calculi $\Gamma((P(D_{p,q}))$ on the quantum discs $D_{p,q}$, we choose the calculi already used in [6]) (see [23,25]). The differential ideal $J(P(D_p) \subset \Omega(P(D_p))$ determining $\Gamma(P(D_p))$ is generated by the elements

$$x(dx) - p^{-1}(dx)x, \qquad x^*(dx^*) - p(dx^*)x^*,$$

$$x(dx^*) - p^{-1}(dx^*)x, \qquad x^*(dx) - p(dx)x^*.$$

Replacing x by y and p by q, one obtains the differential ideal $J(P(D_q)) \subset \Omega(P(D_q))$ determining $\Gamma(P(D_q))$. The corresponding calculus $\Gamma(P(S^2_{pq\phi}))$ on the basis was explicitly described in [6]. Furthermore, we use the right-covariant differential calculus $\Gamma(P(U(1)))$ determined by the right ideal R generated by the element

$$u + \nu u^* - (1+\nu)1,$$

where $0 < \nu \leq 1$. One easily verifies that R fulfills (49). Thus, the differential ideal $J(P(S^1))$ is generated by the sets (50)–(52). Using these generators in the present case one obtains the following relations in $\Gamma_m(P(S^1))$:

$$(du^*)u = u(du^*), \qquad (du^*)u = \nu u(du^*),$$
$$(du^*)u = pu(du^*), \qquad (du^*)u = qu(du^*).$$

Therefore, $du^* = du = 0$ if at least one of the numbers ν, p, q is different from 1. Then, the LC-differential algebra $\Gamma_m(P(S^2_{pq\phi}))$ has the following form:

$$\Gamma^0_m(P(S^2_{pq\phi})) = P(S^2_{pq\phi}), \quad \Gamma^n_m(P(S^2_{pq\phi})) = \Gamma^n(P(D_p)) \oplus \Gamma^n(P(D_q)), \quad n > 0.$$

$\Gamma(P(S^2_{pq\phi}))$ coincides with $\Gamma_m(P(S^2_{pq\phi}))$ for $p \neq q$, and is embedded as a subspace defined by the gluing for $p = q$ (cf. [6]).

Now, we want to construct a connection on the bundle $\mathcal{P}^{(1)}$ which can be regarded as a quantum magnetic monopole with strength $g = -\frac{1}{2}$.

The functionals $\mathcal{X}$ and f on $P(U(1))$ corresponding to the basis element $(u-1)+R \in \ker \varepsilon/R$ are given by

$$\mathcal{X}(u) = 1, \qquad \mathcal{X}(u^*) = -\nu^{-1}, \qquad f(u) = \nu, \qquad f(u^*) = \nu^{-1},$$
$$f(hk) = f(h)f(k), \quad h, k \in P(U(1)), \qquad \mathcal{X}(hk) = \mathcal{X}(h)f(k) + \varepsilon(h)\mathcal{X}(k).$$

$\mathcal{X}$ is a linear basis in the space of functionals annihilating 1 and the right ideal R (see also Appendix A and [27]), i.e. $\mathcal{X}$ is a basis of the ν-deformed Lie algebra corresponding to the differential calculus on $U(1)$. We define the linear maps $A_{l_1} : (U(1)) \to \Gamma(P(D_p))$ and $A_{l_2} : P(U(1)) \to \Gamma(P(D_q))$ corresponding to a left connection on $\mathcal{P}^{(1)}$ by

$$A_{l_1}(h) = \mathcal{X}(h)\tfrac{1}{4}(xdx^* - x^*dx), \quad h \in P(U(1)), \tag{150}$$

$$A_{l_2}(h) = \mathcal{X}(h)\tfrac{1}{4}(y^*dy - ydy^*), \quad h \in P(U(1)). \tag{151}$$

Because of $\mathcal{X}(R) = 0$ and $\mathcal{X}(1) = 0$, A_{l_1} and A_{l_2} fulfill the conditions (70) and (98). Since there is no gluing in $\Gamma^1_m(B)$, the condition (71) is also fulfilled. Therefore, any choice of one-forms to the right of $\mathcal{X}$ gives a connection.

A short calculation shows (see formula (125)) that the linear maps $F_1 : P(U(1)) \to \Gamma^2(P(D_p))$ and $F_2 : P(U(1)) \to \Gamma^2(P(D_q))$ corresponding to the curvature have the following form:

$$F_1(h) = \mathcal{X}(h)\tfrac{1}{4}(1+p)dxdx^* + \sum \mathcal{X}(h_{(1)})\mathcal{X}(h_{(2)})\tfrac{1}{16}(xx^* - px^*x)dxdx^*,$$

$$F_2(h) = -\mathcal{X}(h)\tfrac{1}{4}(1+q)dydy^* + \sum \mathcal{X}(h_{(1)})\mathcal{X}(h_{(2)})\tfrac{1}{16}(yy^* - qy^*y)dydy^*.$$

In the classical case, the local connection forms A_{l_1} and A_{l_2} can be transformed, using suitable local coordinates, from the classical unit discs to the upper and lower hemispheres of the classical S^2. The resulting local connection forms on S^2 just coincide with the well-known magnetic potentials of the Dirac monopole of charge $-\frac{1}{2}$. To explain this we will briefly describe the classical Dirac monopole (see [19]). The classical Dirac monopole is defined on $\mathbb{R}^3 \setminus \{0\}$, which is of the same homotopy type as S^2. The corresponding gauge theory is a $U(1)$ theory, and the Dirac monopole is described as a connection on a $U(1)$ principal fibre bundle over S^2.

Let $\{U_\mathrm{N}, U_\mathrm{S}\}$ be a covering of S^2, where U_N respectively U_S is the closed northern respectively southern hemisphere, $U_\mathrm{N} \cap U_\mathrm{S} = S^1$. One can write U_N and U_S in polar coordinates (up to the poles)

$$U_\mathrm{N} = \left\{(\theta, \phi), \quad 0 < \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi < 2\pi\right\} \cup \{N\},$$
$$U_\mathrm{S} = \left\{(\theta, \phi), \quad \frac{\pi}{2} \leq \theta < \pi, \quad 0 \leq \phi < 2\pi\right\} \cup \{S\}.$$

By

$$g_{12}^{(n)}(\phi) = \exp(\mathrm{i}\, n\phi), \quad 0 \leq \phi < 2\pi, \ \ n \in \mathbb{Z}$$

a family of transition functions $g_{12}^{(n)} : S^1 \to U(1)$, $n \in \mathbb{Z}$ is given. A standard procedure defines a corresponding family of $U(1)$ principal fibre bundles $Q^{(n)}$.

Let $\xi_i : S^1 \to U_i, i = N, S$ be the embedding defined by $\xi_i(\phi) = (\pi/2, \phi)$. A connection on $Q^{(n)}$ is defined by two Lie algebra valued one-forms A_N and A_S fulfilling

$$\xi_\mathrm{N}^*(A_\mathrm{N}) = \xi_\mathrm{S}^*(A_\mathrm{S}) + \mathrm{i}\, n d\phi.$$

The Wu-Yang forms defined by

$$A_\mathrm{N}^{(n)} = \mathrm{i}\tfrac{1}{2} n(1 - \cos\theta) d\phi, \qquad A_\mathrm{S}^{(n)} = -\mathrm{i}\tfrac{1}{2} n(1 + \cos\theta) d\phi$$

fulfill these condition. $A_\mathrm{N}^{(n)}$ and $A_\mathrm{S}^{(n)}$ are vector potentials generating the magnetic field $B = (\frac{1}{2}n)(\vec{r}/|\vec{r}|^3)$, which is interpreted as a monopole of strength $\frac{1}{2}n$.

The classical analogue $\tilde{P}^{(n)}$ of the locally trivial QPFB $\mathcal{P}^{(n)}$ constructed above is a $U(1)$ principal fibre bundle over a space constructed by gluing together two discs along their boundaries. A disc D can be regarded as a subspace of $\mathbb{C}$

$$D := \{x \in \mathbb{C} | xx^* \leq 1\}.$$

The space resulting from gluing together two copies of D over $S^1 = \{x \in D | xx^* = 1\}$ is topologically isomorphic to the sphere S^2. Every $x \in S^1$ has the form $x = \exp(\mathrm{i}\phi)$, $0 \leq \phi < 2\pi$. The classical $U(1)$ bundles $\tilde{P}^{(n)}$ are given by transition functions $\tilde{g}_{12}^{(n)} : S^1 \to U(1)$, which are obtained by $\tau_{12}^{(n)} = (\tilde{g}_{21}^{(n)})^*$ ($*$ means here pull-back) from the above transition functions of QPFB. The exchange of the indices comes from formula (15). One has $\tilde{g}_{12}^{(n)}(\exp(\mathrm{i}\phi)) = \exp(-\mathrm{i}\, n\phi)$, $n \in \mathbb{N}$. Obviously, the $\tilde{P}^{(n)}$ are topologically isomorphic to $Q^{(-n)}$.

The classical analogue of the connection on $\mathcal{P}^{(1)}$ defined above is given by the following one-forms on D (see (150) and (151)):

$$A_1 = \tfrac{1}{4}(xdx^* - x^*dx), \qquad A_2 = \tfrac{1}{4}(x^*dx - xdx^*).$$

Let $\xi : S^1 \to D$ be the embedding. A short calculation shows that A_1 and A_2 fulfill

$$\xi^*(A_1) = \xi^*(A_2) - \mathrm{i}d\phi.$$

Now, one defines the following maps $\eta_{\mathrm{N}} : U_{\mathrm{N}} \setminus \{N\} \to D$ and $\eta_{\mathrm{S}} : U_{\mathrm{S}} \setminus \{S\} \to D$ by

$$\eta_{\mathrm{N}}(\theta, \phi) := \sqrt{1 - \cos\theta}\, \exp(\mathrm{i}\phi), \qquad \eta_{\mathrm{S}}(\theta, \phi) := \sqrt{1 + \cos\theta}\, \exp(\mathrm{i}\phi),$$

and one easily verifies

$$A_{\mathrm{N}}^{(-1)} = \eta_{\mathrm{N}}^*(A_1), \qquad A_{\mathrm{S}}^{(-1)} = \eta_{\mathrm{S}}^*(A_2).$$

6. Final remarks

We have developed the general scheme of a theory of connections on locally trivial QPFB, the main results being the introduction of differential structures on such bundles and the characterization of connections in terms of local connection forms. Here, we make some remarks about questions and problems arising in our context, and about possible future developments.

1. It is very important to look for more examples. Our example of a $U(1)$ bundle over a glued quantum sphere is very similar to the example mentioned in [4] of an $SU_q(2)$ bundle over an analogous glued quantum sphere. Indeed, in [4] another quantum disc is used, which is isomorphic on the C^*-level to the disc used in our paper — both C^*-algebras are isomorphic to the shift algebra. A C^*-version of our bundle can therefore be expected to be isomorphic to the $U(1)$-subbundle of the $SU_q(2)$-bundle of [4] which in turn already determines the latter bundle by the main theorem of [4] (which says that a QPFB with structure group H is determined by a bundle with the classical subgroup of H as structure group). For other examples, one has to look for algebras with a covering (or being a gluing) such that the B_{ij} are "big enough" to allow for nontrivial transition functions $\tau_{ij} : H \to B_{ij} : B_{ij}$ must contain in their centres subalgebras being the homomorphic image of the algebra H. This seems to be possible only if H has nontrivial classical subgroups and B_{ij} contains suitable classical subspaces, as in our example. The following (almost trivial) example of a gluing along two noncommutative parts indicates that one may fall back to a gluing along classical subspaces in many cases. Let $A_1 = C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) = A_2$ be two copies of a quantum sphere being glued together from shift algebras via the symbol map σ, as described in [6]. Then, the gluing $A_1 \oplus_{pr_{1,2}} A_2 := \{((a_1, a_2), (a_1', a_2')) \in A_1 \oplus A_2 | a_2 = a_1'\}$ (gluing of two quantum spheres along hemispheres) is obviously isomorphic to $\{(a_1, a_2, a_3) \in C^*(\mathfrak{S}) \oplus C^*(\mathfrak{S}) \oplus C^*(\mathfrak{S}) | \sigma(a_1) = \sigma(a_2) = \sigma(a_3)\} = C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$. This is a glued quantum sphere with a (quantum disc) membrane inside, glued along the classical subspaces. (This corresponds perfectly to the classical picture of gluing two spheres along hemispheres.)

2. The permanent need to work with covering completions is an unpleasant feature of the theory. It would therefore be very important to find some analogue of algebras of smooth functions in the noncommutative situation which have a suitable class of ideals forming a distributive lattice with respect to $+$ and $\cap$ (cf. [6, Proposition 2]). It is not clear if such a class exists even in classical algebras of differentiable functions.
3. Principal bundles are in the classical case of great importance in topology and geometry. In the above approach, one could, e.g. ask for characteristic classes (trying to generalize the Chern–Weil construction), and for a notion of parallel transport defined by a connection.
4. For locally trivial QPFB, a suitable notion of locally trivial associated quantum vector bundle (QVB) exists ([7]). QVB are defined via cotensor products. One can introduce differential structures on QVB such that one has the usual correspondence between vector valued horizontal forms (of a certain "type") on the QPFB and sections of the associated bundle. To a connection on a QPFB one can associate connections on the corresponding QVB.
5. The notion of gauge transformation in our context is considered in [8]. Gauge transformations are defined as isomorphisms of the left (right) B-module $\mathcal{P}$, with natural compatibility conditions. It turns out that the set of covariant derivatives is invariant under gauge transformations, whereas connections are not always transformed into connections.
6. The relation of our approach to other existing approaches to quantum principal bundles still has to be investigated. In particular, it seems not to be obvious that in our context the canonical map is bijective. This has to be shown as a starting point for a comparison with approaches using Hopf–Galois extensions ([2,14,22,24]). On the other hand, our approach has many similarities to that of [20], which uses methods of sheaf theory. Indeed, every algebra with a covering yields a presheaf of algebras. The underlying topological space is the (finite) index set of the covering, with the discrete topology, and the algebras related to open subsets are gluings restricted to such index subsets. Our choice of tensor products as models of trivial bundles is more special than taking crossed products. However, our approach to differential calculi and connections relies on the simpler structure of tensor products.

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Appendix A

The purpose of this appendix is to collect some results about covariant differential calculi on quantum groups ([5,17,27]) and about coverings and gluings of algebras and differential algebras [6].

A.1. Covariant calculi on Hopf algebras

We freely use standard facts about Hopf algebras. Δ, ε, and S denote comultiplication, counit and antipode, respectively. We use the Sweedler notation (e.g. $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$), and we assume that the antipode is invertible.

A differential algebra over an algebra B is a $\mathbb{N}$-graded algebra $\Gamma(B) = \oplus_{i\in\mathbb{N}} \Gamma^i(B)$, $\Gamma^0((B) = B$, equipped with a differential d, i.e. a graded derivative of degree 1 with $d^2 = 0$. It is called differential calculus if it is generated as an algebra by the db, $b \in B$. A differential ideal of a differential algebra is a d-invariant graded ideal. There is always the universal differential calculus $\Omega(B)$ determined by the property that every differential calculus $\Gamma(B)$ is of the form $\Gamma(B) \simeq \Omega(B)/J(B)$ for some differential ideal $J(B)$.

If two algebras A, B and differential algebras $\Gamma(A)$, $\Gamma(B)$ are given, an algebra homomorphism $\psi : A \to B$ is said to be differentiable with respect to $\Gamma(A)$, $\Gamma(B)$, if there exists a homomorphism $\psi_\Gamma : \Gamma(A) \to \Gamma(B)$ of differential algebras extending ψ (cf. [21]). For $\Gamma(A) = \Omega(A)$ this extension, denoted in this case by $\psi_{\Omega\to\Gamma}$, always exists. If, in addition, $\Gamma(B) = \Omega(B)$, the notation ψ_Ω is used. $J(B) = \ker id_{\Omega\to\Gamma}$ is a differential ideal $J(B) \subset \Omega(B)$ such that $\Gamma(B) = \Omega(B)/J(B)$. $J(B)$ is called the differential ideal corresponding to $\Gamma(B)$.

Now, we list some facts about covariant differential calculi.

Definition 13. A differential calculus $\Gamma(H)$ over a Hopf algebra H is called right-covariant, if $\Gamma(H)$ is a right H comodule algebra with right coaction Δ^Γ such that

$$\Delta^\Gamma(h_0 dh_1 \cdots dh_n) = \Delta(h_0)(d \otimes id) \circ \Delta(h_1) \cdots (d \otimes id) \circ \Delta(h_n). \tag{A.1}$$

$\Gamma(H)$ is called left-covariant, if $\Gamma(H)$ is a left H-comodule algebra with left coaction ${}^\Gamma\Delta$ such that

$$^\Gamma\Delta(h_0 dh_1 \cdots dh_n) = \Delta(h_0)(id \otimes d) \circ \Delta(h_1) \cdots (id \otimes d) \circ \Delta(h_n). \tag{A.2}$$

$\Gamma(H)$ is called bicovariant if it is left- and right-covariant.

Because of the universality property the universal differential calculus over any Hopf algebra is bicovariant. In the sequel, we list some properties of right-covariant differential calculi. The construction of left-covariant differential algebras is analogous.

Let Δ^Ω be the right coaction of the universal differential calculus $\Omega(H)$ and let $\Gamma(H)$ be a differential algebra over the Hopf algebra H. $\Gamma(H)$ is right-covariant if and only if the corresponding differential ideal $J(H) \subset \Omega(H)$ has the property

$$\Delta^\Omega(J(H)) \subset J(H) \otimes H.$$

Let us consider a right-covariant differential calculus $\Gamma(H)$. Let $\Gamma^1_{\rm inv}(H) := \{\gamma \in \Gamma(H) | \Delta^\Gamma(\gamma) = \gamma \otimes 1\}$. There exists a projection $P : \Gamma^1(H) \to \Gamma^1_{\rm inv}(H)$ defined by

$$P(h^0 dh^1) = \sum S^{-1}(h^0_{(2)} h^1_{(2)}) h^0_{(1)} dh^1_{(1)}.$$

Now, one can define a linear map $\eta_\Gamma : H \to \Gamma^1(H)$ by

$$\eta_\Gamma(h) := P(dh) = \sum S^{-1}(h_{(2)})dh_{(1)}.$$

By an easy calculation one obtains the identity $dh = \sum h_{(2)}\eta_\Gamma(h_{(1)})$. The linear map η_Γ has the following properties:

$$\Delta^\Gamma(\eta_\Gamma(h)) = \eta_\Gamma(h) \otimes 1, \qquad \eta_\Gamma(h)k = \sum k_{(2)}(\eta_\Gamma(hk_{(1)}) - \varepsilon(h)\eta_\Gamma(k_{(1)})),$$
$$d\eta_\Gamma(h) = -\sum \eta_\Gamma(h_{(2)})\eta_\Gamma(h_{(1)}).$$

In the case $\Gamma(H) = \Omega(H)$, we use the symbol η_Ω.

The first degrees of right-covariant differential algebras are in one-to-one correspondence to right ideals $R \subset \ker\varepsilon \subset H$ in the following sense. First, if a differential calculus is given, $R := \ker\eta_\Gamma \cap \ker\varepsilon$ is a right ideal with the property $R \subset \ker\varepsilon$, and one can prove that the subbimodule $J^1(H)$ corresponding to $\Gamma^1(H) \cong \Omega^1(H)/J^1(H)$ is generated by the space $\eta_\Omega(R) = \{\sum S^{-1}(r_{(2)})dr_{(1)} | r \in R\}$. On the other hand, every right ideal $R \subset \ker\varepsilon$ defines a right-covariant differential algebra $\Gamma(H) = \Omega(H)/J(H)$, where the differential ideal $J(H) \subset \Omega(H)$ is generated by the set $\eta_\Omega(R)$. Analogously, right ideals $R \subset \ker\varepsilon$ also correspond to left-covariant differential calculi. In this case, the differential ideal $J(H)$ corresponding to R is generated by $\{\sum S(r_{(1)})dr_{(2)} | r \in R\}$. Bicovariant differential calculi are given by right ideals R with the property $\sum S(r_{(1)})r_{(3)} \otimes r_{(2)} \subset H \otimes R \forall r \in R$ (Ad-invariance).

Assuming that $\ker\varepsilon/R$ is finite dimensional one can choose a linear basis $h_i + R$ in $\ker\varepsilon/R$. This leads to a set of functionals $\mathcal{X}_i$ on H annihilating 1 and R such that $\eta_\Gamma(h) = \sum_i \mathcal{X}_i(h)\eta_\Gamma(h_i)$, $h \in H$. The set of the elements $\eta_\Gamma(h_i)$ is a left and right H module basis in $\Gamma^1(H)$, and the set of the $\mathcal{X}_i$ is a linear basis in the space of all functionals annihilating 1 and R. It is obvious that $dh = \sum h_{(2)}\mathcal{X}_i(h_{(1)})\eta_\Gamma(h_i)$. Besides the functionals $\mathcal{X}_i$ the linear basis in $\ker\varepsilon/R$ determines also functionals f_{ij} on H satisfying

$$f_{ij}(1) = \delta_{ij}, \quad f_{ij}(hk) = \sum_l f_{il}(h)f_{lj}(k), \quad \mathcal{X}_i(hk) = \sum_l \mathcal{X}_l(h)f_{li}(k) + \varepsilon(h)\mathcal{X}_i(k).$$

Definition 14. Let A be a vector space and let H be a Hopf algebra such that there exists linear map $\Delta_A : A \to A \otimes H$. Δ_A is called right H-coaction and A is called right H comodule if

$$(\Delta_A \otimes id) \circ \Delta_A = (id \otimes \Delta) \circ \Delta_A, \tag{A.3}$$

$$(id \otimes \varepsilon) \circ \Delta_A = id. \tag{A.4}$$

If A is an algebra and Δ_A is an homomorphism of algebras then A is called a right H comodule algebra. The left coaction is defined analogously.

The definition of covariant differential calculi over Hopf algebras is easily generalized to H comodule algebras.

Definition 15. A differential calculus $\Gamma(A)$ over a right H comodule algebra A is called right-covariant if the right coaction $\Delta_A^\Gamma : \Gamma(A) \to \Gamma(A) \otimes H$ defined by

$$\Delta_A^\Gamma(a_0 da_1 \cdots da_n) = \Delta_A(a_0)(d \otimes id) \circ \Delta_A(a_1) \cdots (d \otimes id) \circ \Delta_A(a_n) \tag{A.5}$$

exists.

The universal calculus $\Omega(A)$ is always right-covariant, with right coaction denoted by Δ_A^Ω. A differential algebra $\Gamma(A)$ over A is right-covariant if and only if $\Delta_A^\Omega(J(A)) \subset J(A) \otimes H$ for the differential ideal $J(A)$ corresponding to $\Gamma(A)$.

A.2. Covering and gluing

Let finite families $(B_i)_{i\in I}$, $(B_{ij})_{(i,j)\in I\times I\setminus D}$, D the diagonal in $I \times I$, $B_{ij} = B_{ji}$, and homomorphisms $\pi_j^i : B_i \to B_{ij}$ be given. Then, the algebra

$$B = \left\{ (b_i)_{i\in I} \in \bigoplus_i B_i \,\middle|\, \pi_j^i(b_i) = \pi_i^j(b_j) \quad \forall i \neq j \right\} =: \bigoplus_{\pi_j^i} B_i$$

is called gluing of the B_i along the B_{ij} by means of the π_j^i. Special cases of gluings arise from coverings. A finite covering of an algebra B is a finite family $(J_i)_{i\in I}$ of ideals in B with $\cap_i J_i = 0$. Taking now $B_i = B/J_i$, $B_{ij} = B/(J_i + J_j)$, $\pi_j^i : B_i \to B_{ij}$ the canonical projections $b + J_i \mapsto b + J_i + J_j$, one can form the gluing

$$B_c = \bigoplus_{\pi_j^i} B_i,$$

which is called the covering completion of B with respect to the covering $(J_i)_{i\in I}$. B is always embedded in B_c via the map $K : b \mapsto (b + J_i)_{i\in I}$. The covering $(J_i)_{i\in I}$ is called complete if K is also surjective, i.e. B is isomorphic to B_c. Every two-element covering is complete, as well as every covering of a C^*-algebra. On the other hand, if $B = \oplus_{\pi_j^i} B_i$ is a general gluing, and $p_i : B \to B_i$ are the restrictions of the canonical projections, then $(\ker p_i)_{i\in I}$ is a complete covering of B.

If $\Gamma(B)$ is a differential algebra, a covering $(J_i)_{i\in I}$ of $\Gamma(B)$ is said to be differentiable if the J_i are differential ideals. A differential algebra $\Gamma(B)$ with differentiable covering $(J_i)_{i\in I}$ is called LC-differential algebra (LC : locally calculus), if the factor differential algebras $\Gamma(B)/J_i$ are differential calculi over B/J_i^0 (J_i^0 the degree zero component of J_i) and $J_i^0 \neq 0$, $\forall i$.

Definition 16. Let $(B, (J_i)_{i\in I})$ be an algebra with complete covering, let $B_i = B/J_i$, let $\pi_i : B \to B_i$ be the natural surjections, and let $\Gamma(B)$ and $\Gamma(B_i)$ be differential calculi such that π_i are differentiable and $(\ker \pi_{i\Gamma})_{i\in I}$ is a covering of $\Gamma(B)$. Then $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ is called adapted to $(B, (J_i)_{i\in I})$.

The following proposition is essential for Definition 4.

Proposition 22. *Let $(B, (J_i)_{i\in I})$ be an algebra with complete covering, and let $\Gamma(B_i)$ be differential calculi over the algebras B_i. Up to isomorphy there exists a unique differential calculus $\Gamma(B)$ such that $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ is adapted to $(B, (J_i)_{i\in I})$.*

As shown in [6], the differential ideal corresponding to $\Gamma(B) = \Omega(B)/J(B)$ is just $J(B) = \cap_{i\in I} \ker \pi_{i\,\Omega\to\Gamma}$.

Finally, there is a proposition concerning the covering completion of adapted differential calculi.

Proposition 23. *Let* $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ *be adapted to* $(B, (J_i)_{i\in I})$*. Then, the covering completion of* $(\Gamma(B), (\ker \pi_{i_\Gamma})_{i\in I})$ *is an LC-differential algebra over* B_{c}.

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